Extended Abstract: Reconsidering Intuitionistic Duality

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1 Introduction

This paper proposes a new syntax and proof system called Dualized Intuitionistic Logic (DIL), for intuitionistic propositional logic with the subtraction operator. Our goal is a conservative extension of standard propositional intuitionistic logic with perfect duality (symmetry) between positive and negative connectives. The proof system should satisfy the following metatheoretic properties: soundness, completeness, cut elimination, and substitution. To our knowledge, no existing system achieves these goals. Substitution is needed for cut elimination, but has posed problems for other systems; for example Crolard develops a complex dependency-tracking calculus to obtain substitution for a constructive type theory with subtraction [3].

In this extended abstract, we describe our work in progress on DIL. We have formulated a dualized syntax and proof system, for which we have proved soundness with respect to a standard Kripke semantics for intuitionistic propositional logic. We have also proved substitution. Regarding completeness, we point out an issue in prior work: we exhibit a semantically valid formula which is provable in DIL but which lacks, under the obvious translation, a cut-free proof in the system SLK[1] of Crolard [2]. This shows that Crolard's system cannot satisfy both cut elimination and completeness.

Our motivation is to obtain a new logical foundation for type theory, with a perfect duality between positive and negative computation. Computational classical type theories (CCTTs) like those proposed in [9] and [4] exhibit such a duality, but are not suitable for computation since they, like other CCTTs, lack the canonicity property: closed terms of type T are not necessarily built with a constructor for T (e.g., pairing for $T \wedge T'$). For other work seeking to support control and canonicity, see [5] and works cited there. Like some CCTTs, DIL is formulated as a sequent calculus. In future work, we intend to use the Curry-Howard isomorphism to develop a Dualized Type Theory based on DIL, where positive assumptions will become input variables and negative ones output variables. Cut will become a control operator much like the μ-operator of Curien and Herbelin's system [4]. We thus propose to explore intuitionistic duality as a basis for constructive control, in contrast to (non-constructive) CCTTs.

But the main benefit we are seeking from a completely dualized system is a uniform simultaneous treatment of induction and coinduction. It is well known that induction and coinduction are duals semantically. While this duality has been considered in CCTT [7], no constructive system with induction and coinduction as duals has yet been proposed. Proof assistants like Coq and Agda have unsatisfactory treatments of (mixed) induction and coinduction (see the discussion in [1]): Coq lacks type preservation in the presence of coinductive types, a serious defect in the system, while Agda restricts how inductive and coinductive types can be nested. Our working hypothesis is that a logical foundation based on intuitionistic duality will allow the semantic duality between induction and coinduction to be expressed in type theory, yielding a solution to the problems with these important features in existing systems. Detailed proofs of all lemmas and theorems below may be found in a companion document on the first author's web site.

2 Syntax of Dualized Intuitionistic Logic (DIL)

$$\begin{array}{lcl} \textit{polarities } p & ::= & + \mid - \\ \textit{formulas } T & ::= & A \mid \langle p \rangle \mid T \rightarrow_p T' \mid T \wedge_p T' \end{array}$$

Figure 1: Syntax of formulas for DIL

The syntax for polarities p and then formulas T of Dualized Intuitionistic Logic (DIL) is given in Figure 1. We write $\bar{p}$ for the opposite polarity from p (so $\bar{+} = -$ and $\bar{-} = +$). The formulas A are drawn from a set $\mathcal{A}$ of atomic propositional formulas. As the semantics below will make precise, the logical constructs above can be identified with standard ones as follows: $\langle + \rangle$ is *True*, $\langle - \rangle$ is *False*, $T \wedge_+ T'$ is $T \wedge T'$, $T \wedge_- T'$ is $T \vee T'$, $T \rightarrow_+ T'$ is $T \rightarrow T'$, and $T \rightarrow_- T'$ is subtraction $T' - T'$ (note the reversed order of subformulas).

3 Kripke Semantics for DIL

We will work with standard Kripke models $(W, \preceq, V)$ (cf. Chapter 7 of [8]), where W is a non-empty set of objects called worlds, $\preceq$ is a preorder on W called the accessibility relation, and V maps each world w in W to a subset of $\mathcal{A}$, namely the atomic formulas which are true in w. As standard, V is required to be monotonic: for all $w \in W$ and $A \in \mathcal{A}$, if $A \in V(w)$, then $A \in V(w')$ for all $w' \succeq w$. Figure 2 defines a semantics $[\![T]\!]_w$ relative to a Kripke model $(W, \preceq, V)$, to interpret a formula T in a world $w \in W$. The semantics exactly follows standard semantics for intuitionistic propositional logic with subtraction (cf. [6]).

Theorem 1 (Monotonicity)**.** *Suppose $w \preceq^p w'$. Then $p[\![T]\!]_w$ implies $p[\![T]\!]_{w'}$.*

Proof. If $p = +$, then the theorem is the usual statement of monotonicity, which we will prove in a moment. If $p = -$, then the theorem says that if $w \succeq w'$, then $\neg[\![T]\!]_w$, implies $\neg[\![T]\!]_{w'}$. But this is equivalent to saying that if $w' \preceq w$, then $[\![T]\!]_{w'}$ implies $[\![t]\!]_w$, which follows from the statement of the theorem when $p = -$.

So suppose $p = +$. The proof is now by induction on T. The cases for $\langle + \rangle$ and $\langle - \rangle$ are trivial, and the cases for $T_1 \wedge_+ T_2$ and $T_1 \wedge_- T_2$ follow directly by the induction hypothesis. So suppose $T \equiv T_1 \rightarrow_+ T_2$, and suppose $[\![T]\!]_w$ and $w \preceq w'$. To show the required $[\![T_1 \rightarrow_+ T_2]\!]_{w'}$, it suffices by the definition of the semantics to assume an arbitrary $w'' \in W$ with $w' \preceq w''$ and $[\![T_1]\!]_{w''}$. We must then show $[\![T_2]\!]_{w''}$. By transitivity of $\preceq$, we have $w \preceq w''$. Unfolding the definition of the semantics, the assumed $[\![T_1 \rightarrow_+ T_2]\!]_w$ becomes:

$$\forall w'.\, w \preceq w' \;\Rightarrow\; [\![T_1]\!]_{w'} \;\Rightarrow\; [\![T_2]\!]_{w''}$$

We can instantiate this with w'', $w \preceq w''$, and $[\![T_1]\!]_{w''}$ to obtain the desired $[\![T_2]\!]_{w''}$.

$$\begin{array}{lcl} [\![A]\!]_w & \Leftrightarrow & A \in V(w) \\ [\![\langle + \rangle]\!]_w & \Leftrightarrow & \textit{true} \\ [\![\langle - \rangle]\!]_w & \Leftrightarrow & \textit{false} \\ [\![T \rightarrow_+ T']\!]_w & \Leftrightarrow & \forall w'.w \preceq w' \Rightarrow [\![T]\!]_{w'} \Rightarrow [\![T']\!]_{w'} \\ [\![T \rightarrow_- T']\!]_w & \Leftrightarrow & \exists w'.w \succeq w' \wedge \neg[\![T]\!]_{w'} \wedge [\![T']\!]_{w'} \\ [\![T \wedge_+ T']\!]_w & \Leftrightarrow & [\![T]\!]_w \wedge [\![T']\!]_w \\ [\![T \wedge_- T']\!]_w & \Leftrightarrow & [\![T]\!]_w \vee [\![T']\!]_w \end{array}$$

Figure 2: Semantics of DIL formulas

$$
\begin{array}{lcl}
\textit{local contexts } \Delta & ::= & \cdot \mid pT, \Delta \\
\textit{modal contexts } \Gamma & ::= & \Delta \mid \Gamma \preceq^p \Delta
\end{array}
$$

Figure 3: Logical contexts for DIL

For the last case, suppose $T \equiv T_1 \rightarrow_- T_2$, and $w \preceq w'$. Unfolding the definition of the semantics, our assumption $[\![T_1 \rightarrow_- T_2]\!]_w$ is equivalent to:

$$\exists w''.\, w \succeq w'' \ \wedge\ \neg[\![T_1]\!]_{w''} \ \wedge\ [\![T_2]\!]_{w''}$$

So assume an arbitrary such w'' satisfying the displayed conjuncts. We must now prove $[\![T_1 \rightarrow_- T_2]\!]_{w'}$, which is equivalent to:

$$\exists w''.\, w' \succeq w'' \ \wedge\ \neg[\![T_1]\!]_{w''} \ \wedge\ [\![T_2]\!]_{w''}$$

Take w'' for this w''. The second two conjuncts are satisfied by our current assumptions, and the first, $w' \succeq w''$, holds by transitivity of $\preceq$ from the assumed facts $w' \succeq w \succeq w''$. □

Relational notation. We write $\preceq^p$ to indicate $\preceq$ if $p = +$, and $\succeq$ if $p = -$.

Paths. We define a bi-directional path π of a Kripke model to be a possibly empty list of worlds $w_1, \cdots, w_n$ such that for all $i \in \{1, \ldots, n-1\}$, we have either $w_i \preceq w_{i+1}$ or $w_i \succeq w_{i+1}$. For such a path, we denote its length (n) by $|\pi|$. We will write $\pi \preceq^p w$ to mean that $\pi = \pi', w'$ for some π' and w', with $w' \preceq^p w$. We do not distinguish a singleton path w from the world w.

4 A Proof System for DIL

In this section we define a proof system for DIL, which we will subsequently justify using the modal semantics defined above. The derivable objects of the proof system are sequents of the form $\Gamma \vdash^p T$, where Γ is a modal context as defined in Figure 3.

4.1 Contexts

Intuitively, a local context Δ describes a world, while a modal context describes a (bi-directional) path. The starting point of the path corresponds to the leftmost local context in the modal context. We treat pT as an abbreviation for local context $pT, \cdot$. We sometimes also view contexts as built from right to left instead of left to right. We concatenate local contexts with Δ, Δ', and modal contexts with $\Gamma \preceq^p \Gamma'$. We do not distinguish a modal context of the form Δ from the local context Δ. The local extension Γ, Δ of modal context Γ by local context Δ is defined as follows:

$$(\Gamma' \preceq^p \Delta'), \Delta = \Gamma \preceq^p (\Delta', \Delta)$$

The local concatenation Γ, Γ' of modal contexts Γ and Γ' is then defined by:

$$\Gamma, (\Delta \preceq^p \Gamma'') = (\Gamma, \Delta) \preceq^p \Gamma''$$

When $\Gamma = \Delta$ or $\Gamma = \Gamma' \preceq^p \Delta$, we call Δ the current local context of Γ. It describes the abstract world w at the end of the abstract path described by Γ. We sometimes refer informally to that w as the current local world of Γ. The length $|\Gamma|$ of modal context Γ is the number of maximal local contexts contained in Γ; so, 1 plus the number of occurrences of $\preceq^+$ or $\preceq^-$. The use of an ordering in the context may also be found in the display calculus δ**BiInt** of Goré [6].

$$\frac{}{\Gamma \ \preceq^{p'} \ \Delta_1, pT, \Delta_2 \vdash^p \ T} \ \textsc{ax} \qquad \frac{\Gamma \vdash^p \ T}{\Gamma \ \preceq^p \ \Delta \vdash^p \ T} \ \textsc{weak} \qquad \frac{}{\Gamma \vdash^p \ \langle p \rangle} \ \textsc{unit}$$

$$\frac{\Gamma \ \preceq^p \ pT \vdash^p \ T'}{\Gamma \vdash^p \ T \to_p T'} \ \textsc{imp} \qquad \frac{\Gamma \vdash^{\bar{p}} \ T \quad \Gamma \vdash^p \ T'}{\Gamma \vdash^p \ T \to_{\bar{p}} T'} \ \textsc{impBar} \qquad \frac{\Gamma \vdash^p \ T_1 \quad \Gamma \vdash^p \ T_2}{\Gamma \vdash^p \ T_1 \wedge_p T_2} \ \textsc{and}$$

$$\frac{\Gamma \vdash^p \ T_1}{\Gamma \vdash^p \ T_1 \wedge_{\bar{p}} T_2} \ \textsc{andBar1} \qquad \frac{\Gamma \vdash^p \ T_2}{\Gamma \vdash^p \ T_1 \wedge_{\bar{p}} T_2} \ \textsc{andBar2} \qquad \frac{\Gamma, \bar{p}\,T \vdash^p \ T' \quad \Gamma, \bar{p}\,T \vdash^{\bar{p}} \ T'}{\Gamma \vdash^p \ T} \ \textsc{cut}$$

Figure 4: Proof Rules for DIL

$$\begin{array}{lcl}
[\![pT, \Delta]\!]_w & = & p \ [\![T]\!]_w \ \wedge \ [\![\Delta]\!]_w \\
[\![\cdot]\!]_w & = & \mathit{true} \\
\\
[\![\Delta]\!]_{\pi,w} & = & [\![\Delta]\!]_w \\
[\![T]\!]_{\pi,w} & = & [\![T]\!]_w \\
\\
[\![\Gamma \ \preceq^p \ \Delta]\!]_{\pi,w} & = & [\![\Gamma]\!]_\pi \ \wedge \ \pi \preceq^p w \ \wedge \ [\![\Delta]\!]_w \\
\\
[\![\Gamma \vdash^p \ T]\!]_\pi & = & [\![\Gamma]\!]_\pi \ \Rightarrow \ p[\![T]\!]_\pi
\end{array}$$

Figure 5: Semantics of Local Contexts and Sequents

4.2 Proof system

Let us write $p\ F$ as meta-notation meaning F if $p \equiv +$ and $\neg F$ if $p \equiv -$. The intuitive meaning of the sequents, which we will make precise in the next section, is: $\Gamma \ \vdash^+ \ T$ iff in the world w at the end of the path determined by Γ, $p[\![T]\!]_w$ holds. The crucial idea of the proof system is to incorporate Theorem 1 (Monotonicity), in the WEAK rule: if we are following edges forward in the accessibility relation, then true formulas will remain true; and dually, if we follow edges backwards, false formulas will remain false.

Figure 4 gives the proof rules for deriving sequents $\Gamma \ \vdash^p \ T$. The rules allow expansion of the context in two different ways. If we expand the context locally, as in the CUT rule, we are adding a new assumption about the current world. If we expand the context modally, as in the IMP rule, we are extending our path π to a new world w' where $\pi \preceq^p w'$, and the assumed formula holds.

5 Semantics and Metatheory for DIL

For purposes of this section, fix an arbitrary Kripke model $(W, \preceq, V)$. Figure 5 defines a semantics for sequents, after first defining several helper predicates: $[\![\Delta]\!]_w$ expresses that world w satisfies local context Δ; $[\![\Delta]\!]_\pi$ expresses that the last world in the path π satisfies Δ; and $[\![\Gamma]\!]_\pi$ expresses that path π satisfies modal context Γ. The interpretation $[\![\Gamma \ \vdash^p \ T]\!]_\pi$ of sequents with respect to a path π is then defined using those predicates: $\Gamma \ \vdash^p \ T$ holds along path π iff assuming π satisfies Γ, then T holds in the last world of π.

Lemma 2 (Local context concatenation)**.** $[\![\Delta', \Delta]\!]_w \Leftrightarrow [\![\Delta']\!]_w \wedge [\![\Delta]\!]_w$

Proof. The proof is by induction on Δ'. For the base case, we have:

$$\begin{array}{ll} [\![\Delta', \Delta]\!]_w & \Leftrightarrow \\ [\![\Delta]\!]_w & \Leftrightarrow \\ \mathit{true} \wedge [\![\Delta]\!]_w & \Leftrightarrow \\ [\![\Delta']\!]_w \wedge [\![\Delta]\!]_w & \end{array}$$

For the step case, suppose $\Delta' = p\,T, \Delta''$. Then we have:

$$\begin{array}{ll} [\![\Delta', \Delta]\!]_w & \Leftrightarrow \\ [\![p\,T, \Delta'', \Delta]\!]_w & \Leftrightarrow \\ p[\![T]\!]_w \wedge [\![\Delta'', \Delta]\!]_w & \Leftrightarrow \\ p[\![T]\!]_w \wedge [\![\Delta'']\!]_w \wedge [\![\Delta]\!]_w & \Leftrightarrow \\ [\![p\,T, \Delta'']\!]_w \wedge [\![\Delta]\!]_w & \Leftrightarrow \\ [\![\Delta']\!]_w \wedge [\![\Delta]\!]_w & \end{array}$$

□

Theorem 3 (Soundness)**.** *If $\Gamma_a \vdash^p T_a$ is derivable using the rules of Figure 4, and if π is a path where $|\pi| = |\Gamma_a|$ (the lengths of the path and the context are the same), then $[\![\Gamma_a \vdash^p T_a]\!]_\pi$.*

Proof. The proof is by induction on the assumed derivation. Since the lengths of the path and the context are the same, and since contexts always have length at least one (since at a minimum, they consist of the empty local context), we will assume below that $\pi = \pi', w$ for some π' and w.

Case:

$$\frac{}{\Gamma \preceq^{p'} \Delta_1, p\,T, \Delta_2 \vdash^p T}\ \text{AX}$$

Assume $[\![\Gamma \preceq^{p'} \Delta_1, p\,T, \Delta_2]\!]_{\pi',w}$. By the definition of the semantics of modal contexts (Figure 5), this assumption implies $[\![\Delta_1, p\,T, \Delta_2]\!]_w$. Now we can apply Lemma 2 twice (once to conclude $[\![\Delta_1, p\,T]\!]_w$ and once more to conclude $[\![p\,T]\!]_w$) to conclude the required $p[\![T]\!]_w$.

Case:

$$\frac{\Gamma \vdash^p T}{\Gamma \preceq^p \Delta \vdash^p T}\ \text{WEAK}$$

To prove the interpretation of the conclusion, first assume $[\![\Gamma \preceq^p \Delta]\!]_{\pi',w}$. By the definition of the semantics of modal contexts, this implies:

- $[\![\Gamma]\!]_{\pi'}$
- $\pi' \preceq^p w$
- $[\![\Delta]\!]_w$

From the IH and the first of the displayed facts, we conclude $p[\![T]\!]_{\pi'}$. Since $|\pi'| = |\Gamma|$, we know $\pi' = \pi'', w'$ for some π'' and w', where $p[\![T]\!]_{w'}$. The second of the displayed facts then implies that $w' \preceq^p w$. We deduce by Theorem 1 (Monotonicity) that $p[\![T]\!]_w$. This implies $p[\![T]\!]_\pi$ as required.

Case:

$$\frac{}{\Gamma \vdash^p \langle p \rangle}\ \text{UNIT}$$

We must assume $[\![\Gamma]\!]_\pi$, and prove $p[\![\langle p \rangle]\!]$. If $p = +$, the latter is equivalent to *true*. If $p = -$, it is equivalent to $\neg$*false*. In either case, the interpretation of the conclusion holds.

Case:

$$\frac{\Gamma \preceq^p pT \vdash^p T'}{\Gamma \vdash^p T \rightarrow_p T'} \text{ IMP}$$

Assume $[\![\Gamma]\!]_{\pi',w}$. We now case split on p. If $p = +$, then we must prove $[\![T \rightarrow_+ T']\!]_w$. For this, it suffices to assume arbitrary w' with $w \preceq w'$ and $[\![T]\!]_{w'}$, and prove $[\![T']\!]_{w'}$. By the IH, we may conclude that $[\![\Gamma \preceq^+ +T]\!]_{\pi,w'}$ implies the required $[\![T']\!]_{w'}$. By the definition of the semantics of modal contexts, we know that $[\![\Gamma \preceq^p pT]\!]_{\pi,w'}$ is equivalent to the conjunction of:

- $[\![\Gamma]\!]_\pi$
- $\pi \preceq w'$
- $[\![+T]\!]_{w'}$

But all the displayed facts hold in this case: we are assuming $[\![\Gamma]\!]_\pi$; $\pi \preceq w'$ follows because $\pi = \pi', w$ and $w \preceq w'$; and we are assuming $[\![T]\!]_{w'}$.

Suppose now that $p = -$. So we must prove $\neg[\![T \rightarrow_- T']\!]_w$. By the semantics of formulas (Figure 2), this is equivalent to

$$\neg(\exists w'.\ w' \preceq w \ \wedge\ \neg[\![T]\!]_{w'} \ \wedge\ [\![T']\!]_{w'}$$

By standard logical equivalences, this is equivalent to

$$\forall w'.\ w' \preceq w \ \Rightarrow\ \neg[\![T]\!]_{w'} \ \Rightarrow\ \neg[\![T']\!]_{w'}$$

Some assume w' with $w' \preceq w$ and $\neg[\![T]\!]_{w'}$. By the IH, we may conclude that $[\![\Gamma \preceq^- -T]\!]_{\pi,w'}$ implies the required $\neg[\![T']\!]_{w'}$. As above, we reason that $[\![\Gamma \preceq^- -T]\!]_{\pi,w'}$ is equivalent to the conjunction of:

- $[\![\Gamma]\!]_\pi$
- $\pi \preceq^- w'$
- $[\![-T]\!]_{w'}$

As in the case where $p = +$, we have all these facts. We are assuming $[\![\Gamma]\!]_\pi$ and $\neg[\![T]\!]_{w'}$. We have $\pi \preceq^- w'$ because $\pi = \pi', w$ and $w \succeq w'$.

Case:

$$\frac{\begin{array}{c}\Gamma \vdash^{\bar{p}} T \\ \Gamma \vdash^p T'\end{array}}{\Gamma \vdash^p T \rightarrow_{\bar{p}} T'} \text{ IMPBAR}$$

Assume $[\![\Gamma]\!]_{\pi,w}$. Now we case split on p. If $p = +$, then our goal is to conclude $[\![T \rightarrow_- T']\!]_w$. By the definition of the semantics of formulas, we must exhibit some $w' \preceq w$ such that $\neg[\![T]\!]_{w'}$ and $[\![T']\!]_{w'}$. By reflexivity of $\preceq$, we may take w for w'. We then have the required facts for this choice of w' by the IH. Now suppose $p = -$. We must prove $\neg[\![T \rightarrow_+ T']\!]_w$. For this, it suffices to exhibit some $w' \succeq w$ with $[\![T]\!]_{w'}$ and $\neg[\![T']\!]_{w'}$. Again, we may take w for w', and we then have these facts by the IH.

Case:

$$\frac{\begin{array}{c}\Gamma \vdash^p T_1 \\ \Gamma \vdash^p T_2\end{array}}{\Gamma \vdash^p T_1 \wedge_p T_2} \text{ AND}$$

This case follows easily by the IH, using the semantics of formulas and case-splitting on whether $p = +$ or $p = -$. The cases for ANDBAR1 and ANDBAR2 are similar, so we omit them.

Case:

$$\frac{\Gamma, \bar{p}\,T \vdash^{p}\ T' \qquad \Gamma, \bar{p}\,T \vdash^{\bar{p}}\ T'}{\Gamma \vdash^{p}\ T}\ \textsc{cut}$$

Assume $[\![\Gamma]\!]_{\pi',w}$. Let us case split on whether or not $p[\![T]\!]_w$. If it does, then we are done. If not, we have $[\![\Gamma, \bar{p}\,T]\!]_\pi$, by the following argument. If $\Gamma = \Delta$ for some Δ, then we have $[\![\Delta, \bar{p}\,T]\!]_{\pi',w}$ by Lemma 2, since we have $[\![\Delta]\!]_{\pi',w}$ and $\bar{p}[\![T]\!]_w$. Similarly, if $\Gamma = \Gamma' \ \preceq^{p'}\ \Delta$ for some Γ', p', and Δ, then we must show $[\![\Gamma' \ \preceq^{p'}\ (\Delta, \bar{p}\,T)]\!]_{\pi',w}$. For this, it suffices by the definition of the semantics of modal contexts to show:

- $[\![\Gamma']\!]_{\pi'}$
- $\pi' \preceq^{p'} w$
- $[\![\Delta, \bar{p}\,T]\!]_w$

The first two of these follow directly from our assumption $[\![\Gamma' \ \preceq^{p'}\ \Delta]\!]_{\pi',w}$. The last follows as in the case where $\Gamma = \Delta$.

Now that we know $[\![\Gamma, \bar{p}\,T]\!]_\pi$, we can conclude both $p[\![T']\!]_\pi$ and $\bar{p}[\![T']\!]_\pi$ by the IH. These are contradictory, showing that $\bar{p}[\![T]\!]_w$ is impossible.

□

Lemma 4 (Local weakening)**.** *The following rule is admissible:*

$$\frac{\Gamma_1, \Gamma_2 \vdash^{p}\ T}{\Gamma_1, \Delta, \Gamma_2 \vdash^{p}\ T}\ \textsc{localWeak}$$

Proof. The proof is by induction on the structure of the assumed derivation.

Case:

$$\frac{}{\Gamma \ \preceq^{p'}\ \Delta_1, pT, \Delta_2 \vdash^{p}\ T}\ \textsc{ax}$$

We need to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^{p}\ T$. We know

$$\Gamma_1, \Gamma_2 = \Gamma \ \preceq^{p}\ \Delta_1, pT, \Delta_2 \tag{1}$$

We will do a case split on the form of Γ_2. Let us consider the case when Γ_2 is a local context, say Δ_2'. This implies that Γ_1 must contain $\preceq^{p}$ in order for the equality (1) to hold. This implies that $\Gamma_1 = \Gamma \ \preceq^{p}\ \Delta_1'$, where

$$\Delta_1', \Delta_2' = \Delta_1, pT, \Delta_2 \tag{2}$$

We must now consider the following cases depending on where pT falls in equality relationship (2). Let us consider the case when, $\Delta_1' = \Delta_1, pT, \Delta_2''$. This implies that $\Gamma_1, \Delta', \Gamma_2 = \Gamma \ \preceq^{p}\ \Delta_1, pT, \Delta_2'', \Delta', \Delta_2'$ which means we can make the following inference:

$$\frac{}{\Gamma \ \preceq^{p}\ \Delta_1, pT, \Delta_2'', \Delta', \Delta_2' \vdash^{p}\ T}\ \textsc{ax}$$

Let us consider the case when $\Delta_2' = \Delta_1'', pT, \Delta_2$. This implies that $\Gamma_1, \Delta', \Gamma_2 = \Gamma \ \preceq^{p}\ \Delta_1', \Delta', \Delta_1'', pT, \Delta_2$ which means that we can make the following inference:

$$\frac{}{\Gamma \ \preceq^{p}\ \Delta_1', \Delta', \Delta_1'', pT, \Delta_2 \vdash^{p}\ T}\ \textsc{ax}$$

This covers the case where Γ_2 is some local context, let us now consider when $\Gamma_2 = \Gamma' \ \preceq^{p''}\ \Delta''$, but by (1) we know that $p'' = p$ and $\Delta'' = \Delta_1, pT, \Delta_2$. This implies that $\Gamma_1, \Delta', \Gamma_2 = \Gamma_1, \Delta', \Gamma' \ \preceq^{p}$ Δ_1, pT, Δ_2 which means we can make the following inference:

$$\frac{}{\Gamma_1, \Delta', \Gamma' \preceq^p \Delta_1, pT, \Delta_2 \vdash^p T} \text{AX}$$

Case:

$$\frac{\Gamma \vdash^p T}{\Gamma \preceq^p \Delta \vdash^p T} \text{WEAK}$$

We need to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^p T$. We know

$$\Gamma_1, \Gamma_2 = \Gamma \preceq^p \Delta \tag{3}$$

We will need to do a case split on the form of Γ_2.
Let Γ_2 be a local context, say Δ_2. Then we know that $\Gamma_1 = \Gamma \preceq^p \Delta_1$ by (3). This implies that $\Gamma_1, \Delta', \Gamma_2 = \Gamma \preceq^p \Delta_1, \Delta', \Delta_2$ which means that the following inference still holds.

$$\frac{\Gamma \vdash^p T}{\Gamma \preceq^p \Delta_1, \Delta', \Delta_2 \vdash^p T} \text{WEAK}$$

Let us now consider the case when $\Gamma_2 = \Gamma' \preceq^{p'} \Delta''$, by (3) we know that $p' = p$ and that $\Delta'' = \Delta$. This implies that $\Gamma_1, \Delta', \Gamma_2 = \Gamma_1, \Delta', \Gamma' \preceq^p \Delta$ and that $\Gamma = \Gamma_1, \Gamma'$. The following derivation shows that $\Gamma_1, \Delta', \Gamma_2 \vdash^p T$ holds.

$$\frac{\dfrac{\Gamma_1, \Gamma' \vdash^p T}{\Gamma_1, \Delta', \Gamma' \vdash^p T} \text{IH}}{\Gamma_1, \Delta', \Gamma' \preceq^p \Delta \vdash^p T} \text{WEAK}$$

Case:

$$\frac{}{\Gamma \vdash^p \langle p \rangle} \text{UNIT}$$

We know that $\Gamma_1, \Delta', \Gamma_2 \vdash^p \langle p \rangle$ holds as unit always holds regardless of the context.

Case:

$$\frac{\Gamma \preceq^p pT \vdash^p T'}{\Gamma \vdash^p T \to_p T'} \text{IMP}$$

We know that $\Gamma_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^p T \to_p T'$.

$$\frac{\dfrac{\Gamma_1, \Gamma_2 \preceq^p pT \vdash^p T'}{\Gamma_1, \Delta', \Gamma_2 \preceq^p pT \vdash^p T'} \text{IH}}{\Gamma_1, \Delta', \Gamma_2 \vdash^p T \to_p T'} \text{IMP}$$

Case:

$$\frac{\begin{array}{c}\Gamma \vdash^{\bar{p}} T \\ \Gamma \vdash^p T'\end{array}}{\Gamma \vdash^p T \to_{\bar{p}} T'} \text{impBar}$$

We know that $\Gamma_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^p T \to_{\bar{p}} T'$

$$\dfrac{\dfrac{\Gamma_1, \Gamma_2 \vdash^{\bar{p}} \ T}{\Gamma_1, \Delta', \Gamma_2 \vdash^{\bar{p}} \ T}\ \text{IH} \quad \dfrac{\Gamma_1, \Gamma_2 \vdash^{p} \ T'}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T'}\ \text{IH}}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T \rightarrow_{\bar{p}} T'}\ \textsc{impBar}$$

Case:

$$\dfrac{\begin{array}{c}\Gamma \vdash^{p} \ T_1 \\ \Gamma \vdash^{p} \ T_2\end{array}}{\Gamma \vdash^{p} \ T_1 \wedge_p T_2}\ \textsc{and}$$

We know that $\Gamma_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T \wedge_p T'$.

$$\dfrac{\dfrac{\Gamma_1, \Gamma_2 \vdash^{p} \ T}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T}\ \text{IH} \quad \dfrac{\Gamma_1, \Gamma_2 \vdash^{p} \ T'}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T'}\ \text{IH}}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T \wedge_p T'}\ \textsc{and}$$

Case:

$$\dfrac{\Gamma \vdash^{p} \ T_1}{\Gamma \vdash^{p} \ T_1 \wedge_{\bar{p}} T_2}\ \textsc{andBar1}$$

We know that $\Gamma_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T \wedge_{\bar{p}} T'$.

$$\dfrac{\dfrac{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T}{\Gamma_1, \Gamma_2 \vdash^{p} \ T}\ \text{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} \ T \wedge_{\bar{p}} T'}\ \textsc{andBar1}$$

Case:

$$\dfrac{\Gamma \vdash^{p} \ T_2}{\Gamma \vdash^{p} \ T_1 \wedge_{\bar{p}} T_2}\ \textsc{andBar2}$$

We know that $\Gamma_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T \wedge_{\bar{p}} T'$.

$$\dfrac{\dfrac{\Gamma_1, \Gamma_2 \vdash^{p} \ T'}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T'}\ \text{IH}}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T \wedge_{\bar{p}} T'}\ \textsc{andBar2}$$

Case:

$$\dfrac{\begin{array}{c}\Gamma, \bar{p}\, T \vdash^{p} \ T' \\ \Gamma, \bar{p}\, T \vdash^{\bar{p}} \ T'\end{array}}{\Gamma \vdash^{p} \ T}\ \textsc{cut}$$

We know that $\Gamma_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T$.

$$\dfrac{\dfrac{\Gamma_1, \Gamma_2, \bar{p}\, T \vdash^{p} \ T'}{\Gamma_1, \Delta', \Gamma_2, \bar{p}\, T \vdash^{p} \ T'}\ \text{IH} \quad \dfrac{\Gamma_1, \Gamma_2, \bar{p}\, T \vdash^{\bar{p}} \ T'}{\Gamma_1, \Delta', \Gamma_2, \bar{p}\, T \vdash^{\bar{p}} \ T'}\ \text{IH}}{\Gamma_1, \Delta', \Gamma_2 \vdash^{p} \ T}\ \textsc{Cut}$$

$\square$

Theorem 5 (Substitution)**.** *If* $\Gamma_1, p_1\, T_1, \Gamma_2 \vdash^{p_2} \ T_2$ *and* $\Gamma_1 \vdash^{p_1} \ T_1$*, then* $\Gamma_1, \Gamma_2 \vdash^{p_2} \ T_2$*.*

Proof. The proof is by induction on the structure of the first assumed derivation.

Case:

$$\frac{}{\Gamma \preceq^{p'} \Delta_1, p\,T, \Delta_2 \vdash^{p} T}\ \textsc{ax}$$

We need to show that $\Gamma_1, \Gamma_2 \vdash^{p} T$ holds. We know

$$\Gamma_1, p_1\,T_1, \Gamma_2 = \Gamma \preceq^{p} \Delta_1, p\,T, \Delta_2 \tag{4}$$

We now case split on the form of Γ_2. Suppose Γ_2 is some local context Δ_2'. We know Γ_1 cannot also be a local context, because then the left-hand side of (4) would not contain $\preceq^{p}$, as required by the right-hand side. So Γ_1 must equal $\Gamma \preceq^{p'} \Delta_1'$ for some Δ_1' where by (4) we have

$$\Delta_1', p_1\,T_1, \Delta_2' = \Delta_1, p\,T, \Delta_2 \tag{5}$$

Now we case split on whether or not $\Delta_1' = \Delta_1$. If it does, then by (5) we must also have $p_1\,T_1 = p\,T$ and $\Delta_2' = \Delta_2$. So to obtain the desired $\Gamma_1, \Gamma_2 \vdash^{p} T$, we can use:

$$\frac{\Gamma_1 \vdash^{p_1} T_1}{\Gamma_1, \Delta_2' \vdash^{p_1} T_1}\ \textsc{localWeak}$$

This derives the correct conclusion because $\Gamma_2 = \Delta_2'$ and $p_1\,T_1 = p\,T$.

Now suppose $\Delta_1' \neq \Delta_1$. This implies that Δ_1', Δ_2' still contains $p\,T$, by (4). So we can obtain the desired conclusion this way:

$$\frac{}{\Gamma \preceq^{p'} \Delta_1', \Delta_2' \vdash^{p} T}\ \textsc{ax}$$

Now let us consider the case when $\Gamma_2 = \Gamma_2' \preceq^{p''} \Delta_2'$ for some Γ_2', p'', and Δ_2'. But then by (4), $\Delta_2' = \Delta_1, p\,T, \Delta_2$ and $p'' = p'$, and we can derive the desired conclusion this way:

$$\frac{}{\Gamma_1, \Gamma_2' \preceq^{p'} \Delta_1, p\,T, \Delta_2 \vdash^{p} T}\ \textsc{ax}$$

Case:

$$\frac{\Gamma \vdash^{p} T}{\Gamma \preceq^{p} \Delta \vdash^{p} T}\ \textsc{weak}$$

We know

$$\Gamma_1, p_1\,T_1, \Gamma_2 = \Gamma \preceq^{p} \Delta \tag{6}$$

We case split on the form of Γ_2. Suppose $\Gamma_2 = \Gamma_2' \preceq^{p} \Delta$. Then (6) implies $\Gamma = \Gamma_1, p_1\,T_1, \Gamma_2'$, and we have the following derivation:

$$\frac{\dfrac{\Gamma_1, p_1\,T_1, \Gamma_2' \vdash^{p} T}{\Gamma_1, \Gamma_2' \vdash^{p} T}\ \text{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} T}\ \textsc{weak}$$

Now suppose $\Gamma_2 = \Delta_2$. Then by (6), Γ_1 must equal $\Gamma \preceq^{p} \Delta_1$, for some Δ_1 where $\Delta = \Delta_1, p_1\,T_1, \Delta_2$. This implies $\Gamma_1, \Gamma_2 = \Gamma \preceq^{p} \Delta_1, \Delta_2$ so we can use:

$$\frac{\Gamma \vdash^{p} T}{\Gamma \preceq^{p} \Delta_1, \Delta_2 \vdash^{p} T}\ \textsc{weak}$$

Case:

$$\frac{}{\Gamma \vdash^{p} \langle p \rangle}\ \textsc{unit}$$

We need to show that $\Gamma_1, \Gamma_2 \vdash^{p} \langle p \rangle$. Since unit holds regardless of what the context is, the result trivially follows.

Case:

$$\frac{\Gamma \preceq^{p} pT \vdash^{p} T'}{\Gamma \vdash^{p} T \rightarrow_{p} T'}\ \textsc{imp}$$

We know that $\Gamma_1, p_1\, T_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Gamma_2 \vdash^{p} T \rightarrow_{p} T'$.

$$\frac{\dfrac{\Gamma_1, p_1\, T_1, \Gamma_2 \preceq^{p} pT \vdash^{p} T'}{\Gamma_1, \Gamma_2 \preceq^{p} pT \vdash^{p} T'}\ \mathrm{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} T \rightarrow_{p} T'}\ \textsc{imp}$$

Case:

$$\frac{\Gamma \vdash^{\bar{p}} T \qquad \Gamma \vdash^{p} T'}{\Gamma \vdash^{p} T \rightarrow_{\bar{p}} T'}\ \textsc{impBar}$$

We know that $\Gamma_1, p_1\, T_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Gamma_2 \vdash^{p} T \rightarrow_{\bar{p}} T'$

$$\frac{\dfrac{\Gamma_1, p_1\, T_1, \Gamma_2 \vdash^{\bar{p}} T}{\Gamma_1, \Gamma_2 \vdash^{\bar{p}} T}\ \mathrm{IH} \quad \dfrac{\Gamma_1, p_1\, T_1, \Gamma_2 \vdash^{p} T'}{\Gamma_1, \Gamma_2 \vdash^{p} T'}\ \mathrm{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} T \rightarrow_{\bar{p}} T'}\ \textsc{impBar}$$

Case:

$$\frac{\Gamma \vdash^{p} T_1 \qquad \Gamma \vdash^{p} T_2}{\Gamma \vdash^{p} T_1 \wedge_{p} T_2}\ \textsc{and}$$

We know that $\Gamma_1, p_1\, T_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Gamma_2 \vdash^{p} T \wedge_{p} T'$.

$$\frac{\dfrac{\Gamma_1, p_1\, T_1, \Gamma_2 \vdash^{p} T}{\Gamma_1, \Gamma_2 \vdash^{p} T}\ \mathrm{IH} \quad \dfrac{\Gamma_1, p_1\, T_1, \Gamma_2 \vdash^{p} T'}{\Gamma_1, \Gamma_2 \vdash^{p} T'}\ \mathrm{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} T \wedge_{p} T'}\ \textsc{and}$$

Case:

$$\frac{\Gamma \vdash^{p} T_1}{\Gamma \vdash^{p} T_1 \wedge_{\bar{p}} T_2}\ \textsc{andBar1}$$

We know that $\Gamma_1, p_1\, T_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Gamma_2 \vdash^{p} T \wedge_{\bar{p}} T'$.

$$\dfrac{\dfrac{\Gamma_1, p_1\, T_1, \Gamma_2 \vdash^{p} \;\; T}{\Gamma_1, \Gamma_2 \vdash^{p} \;\; T}\ \text{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} \;\; T \wedge_{\bar{p}} T'}\ \textsc{andBar1}$$

<u>**Case:**</u>

$$\dfrac{\Gamma \vdash^{p} \;\; T_2}{\Gamma \vdash^{p} \;\; T_1 \wedge_{\bar{p}} T_2}\quad \textsc{andBar2}$$

We know that $\Gamma_1, p_1\, T_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Gamma_2 \vdash^{p} \;\; T \wedge_{\bar{p}} T'$.

$$\dfrac{\dfrac{\Gamma_1, p_1\, T_1, \Gamma_2 \vdash^{p} \;\; T'}{\Gamma_1, \Gamma_2 \vdash^{p} \;\; T'}\ \text{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} \;\; T \wedge_{\bar{p}} T'}\ \textsc{andBar2}$$

<u>**Case:**</u>

$$\dfrac{\begin{array}{c}\Gamma, \bar{p}\, T \vdash^{p} \;\; T' \\ \Gamma, \bar{p}\, T \vdash^{\bar{p}} \;\; T'\end{array}}{\Gamma \vdash^{p} \;\; T}\quad \textsc{cut}$$

We know that $\Gamma_1, p_1\, T_1, \Gamma_2 = \Gamma$. The following derivation suffices to show that $\Gamma_1, \Gamma_2 \vdash^{p} \;\; T$.

$$\dfrac{\dfrac{\Gamma_1, p_1\, T_1, \Gamma_2, \bar{p}\, T \vdash^{p} \;\; T'}{\Gamma_1, \Gamma_2, \bar{p}\, T \vdash^{p} \;\; T'}\ \text{IH} \quad \dfrac{\Gamma_1, p_1\, T_1, \Gamma_2, \bar{p}\, T \vdash^{\bar{p}} \;\; T'}{\Gamma_1, \Gamma_2, \bar{p}\, T \vdash^{\bar{p}} \;\; T'}\ \text{IH}}{\Gamma_1, \Gamma_2 \vdash^{p} \;\; T}\ \textsc{Cut}$$

□

6 Possible Incompleteness of Other Logics

Consider the following formula

$$A \to_{+} (A \to_{-} A \to_{+} \langle - \rangle) \to_{-} \langle + \rangle \qquad (\star_1).$$

Using the usual translation to classical logic one will see that this formula is an embedding of the law of excluded middle into intuitionistic logic. It is valid with respect to the semantics given in Section 3.

Lemma 6. *Suppose $M = \langle W, \preceq, V\rangle$ is a Kripke model, then for all worlds $w \in W$ we have $[\![A \to_{+} (A \to_{-} A \to_{+} \langle - \rangle) \to_{-} \langle + \rangle]\!]_w$.*

Proof. By definition we must show that $\forall w_1.(w \preceq w_1$ and $[\![A]\!]_{w_1}) \Rightarrow [\![(A \to_{-} (A \to_{+} \langle - \rangle)) \to_{-} \langle + \rangle]\!]_{w_1}$. Now suppose $w_1 \in W$ such that $w \preceq w_1$ and $[\![A]\!]_{w_1}$. Then we must show $\exists w_2.w_2 \preceq w_1$ and $\neg[\![A \to_{-} (A \to_{+} \langle - \rangle)]\!]_{w_2}$ and $[\![\langle + \rangle]\!]_{w_2}$. Take w_1 for w_2. Clearly, $[\![\langle + \rangle]\!]_{w_1}$ holds. To show $\neg[\![A \to_{-} (A \to_{+} \langle - \rangle)]\!]_{w_1}$ we must show $\forall w_3.w_3 \preceq w_1$ and $\neg[\![A]\!]_{w_3} \Rightarrow \neg[\![A \to_{+} \langle - \rangle]\!]_{w_3}$. So assume $w_3 \in W$, $w_3 \preceq w_1$, and $\neg[\![A]\!]_{w_3}$. Then we must show $\neg[\![A \to_{+} \langle - \rangle]\!]_{w_3}$. It suffices to show $\exists w_4.w_3 \preceq w_4$ and $[\![A]\!]_{w_4}$ and $\neg[\![\langle - \rangle]\!]_{w_4}$. Take w_1 for w_4. Clearly, $\neg[\![\langle - \rangle]\!]_{w_1}$, and by assumption we have $[\![A]\!]_{w_1}$. Therefore, $[\![A \to_{+} (A \to_{-} A \to_{+} \langle - \rangle) \to_{-} \langle + \rangle]\!]_w$. □

In addition, this formula has an easy cut-free derivation (elided) in DIL.

In [2], Crolard defines an intuitionistic logic with subtraction called SLK[1] which he states is complete with respect to the standard Kripke semantics for intuitionistic logic. This system uses the following forms of implication-left and subtraction-right rules, where the opposite context is required to be empty:

$$\frac{\Gamma, B \vdash A}{\Gamma \vdash B \Rightarrow A} \qquad \frac{A \vdash \Delta, B}{A - B \vdash \Delta}$$

We show next that the formula $(\star_1)$ has no cut-free proof in SLK[1]. We first must translate $(\star_1)$ into the language of SLK[1]. Its equivalent form is the following:

$$A \to (True - ((A \to False) - A)) \quad (\star_2).$$

Then using Crolard's definitions $(\star_2)$ has a shorter form $A \to \sim(\neg A - A)$.

Theorem 7. *The formula $A \to (True - ((A \to False) - A))$ has no cut-free proof in* SLK[1].

Proof. We show that the sequent $\cdot \vdash A \to (True - ((A \to False) - A))$ is not derivable using the rules of SLK[1] without cut. Its derivation must begin with the following:

$$\frac{A \vdash True - ((A \to False) - A)}{\cdot \vdash A \to (True - ((A \to False) - A))}$$

At this point we have a few choices. The only rules we could make progress with are the weakening rules, contraction rules, or the right subtraction rule. The right weakening rule would not work, applying the left weakening rule would remove the hypothesis A, which is needed. The right contraction rule would not result in any progress. Now the left contraction rules may be of use. Applying it results in the following derivation.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash True} \qquad A, (A \to False) - A \vdash \cdot}{A, A \vdash True - ((A \to False) - A)}}{A \vdash True - ((A \to False) - A)}}{\cdot \vdash A \to (True - ((A \to False) - A))}$$

The next rule we need to be able to apply is the inference rule for subtraction on the left. However, that rule restricts the left side of the sequent in the conclusion to a single formula. Here we have the hypothesis A in context. So the only rule we could apply to make progress is the inference rules for weakening on the left.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash True} \qquad \cfrac{\cfrac{A \to False \vdash A}{(A \to False) - A \vdash \cdot}}{A, (A \to False) - A \vdash \cdot}}{A, A \vdash True - ((A \to False) - A)}}{A \vdash True - ((A \to False) - A)}}{\cdot \vdash A \to (True - ((A \to False) - A))}$$

We can see that we will never be able to prove A from $\neg A$. Using the inference rule for subtraction on the right directly instead of using contraction first results in the same failure. In fact no matter what we will end up trying to prove A from $\neg A$, because of the restriction on the inference rule for subtraction on the left. Therefore there is no cut free derivation of this formula in SLK[1]. $\square$

The previous theorem shows that SLK^1 cannot satisfy both completeness (which follows directly from Theorem 2.4.3 and Proposition 4.4.1 of [2]) and cut elimination. It is unclear if the previous formula is derivable in SLK^1 if cut is used. Note that Goré's δ**BiInt** [6] does not have any restriction corresponding to that in the implication-right and subtraction-left rules of SLK^1, so we conjecture this formula is derivable in δ**BiInt** (due to lack of experience with display calculi, we have not been able to confirm this yet).

7 Conclusion

We have proposed Dualized Intuitionistic Logic (DIL), which is sound with respect to a standard Kripke semantics for propositional intuitionistic logic with subtraction, and has the substitution property. The crucial idea is to use modal contexts Γ to describe bi-directional paths, and incorporate monotonicity in the proof system. We plan to complete the metatheoretic analysis of DIL, in particular completeness and cut elimination. *Axiom cuts*, where one premise is a weakening (iterated application of WEAK) of AX cannot be eliminated, but we conjecture that all other cuts can be. We then plan to develop DIL into a Dualized Type Theory.

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