CSCI3390-First Test Solutions

February 25, 2016

In this test you never have to give a low-level description of a Turing machine complete with states and transitions (like the kind given in Figure 1). In 1(e) and 3 you are asked to give *high-level* descriptions: these are written in language like, 'the machine scans its input left-to-right, replacing all occurrences of a by X and then writes aaa at the right end of the tape'.

In Problem 2, you should give the answers as informal descriptions of algorithms; you do not need to talk about Turing machines at all.

While Problem 4 refers to the construction in Problem 3, you can do Problem 4 even if you could not carry out the construction.

1 A Turing Machine

(30 points, 10 points for parts (b) and (e) and 5 points for each remaining part) The Turing machine $\mathcal{M}$ pictured in the figure recognizes the language $L \subseteq \{a, b\}^*$ consisting of all the strings in which the number of a's is *less than or equal to* the number of b's.

(a) Let Q be the set of the states of the machine and Γ the tape alphabet. Let

$$\delta : (Q - \{\text{accept}, \text{reject}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

be the next-state function. What are the values of $\delta(1, X)$ and $\delta(1, a)$?

(b) Show the run of the machine (the complete configurations) on the inputs bab and a for ten steps, or until the machine halts, whichever comes first.

(c) Does $\mathcal{M}$ *decide* the language L? Explain *briefly* (a sentence will do).

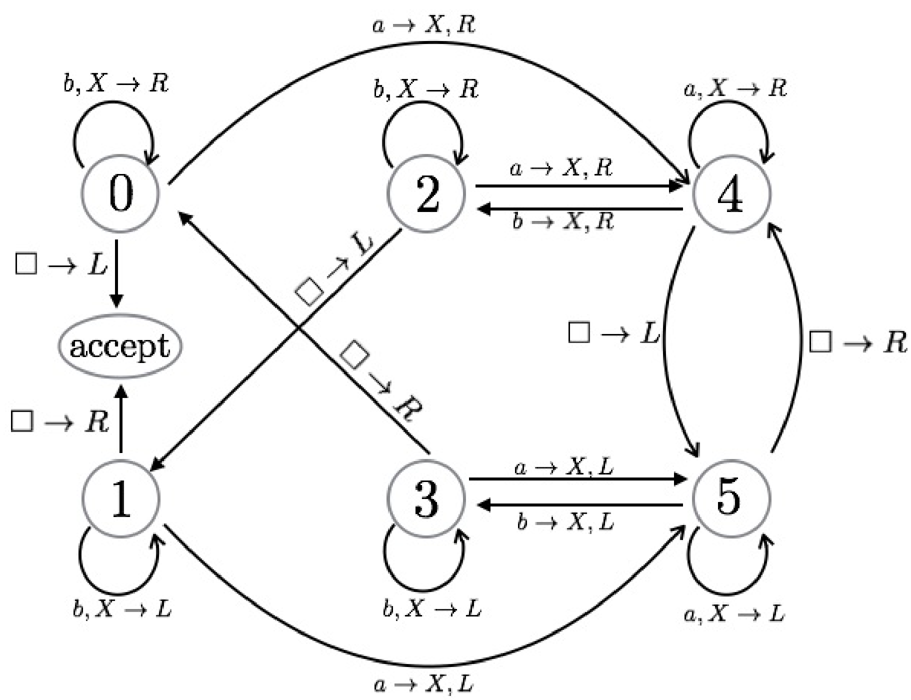

Figure 1: The Turing machine for Problem 1. The initial state is 0.

(d) Is the language L decidable? Explain *briefly.*

(e) On inputs of the form $a^n b^n$, $\mathcal{M}$ takes roughly $4n^2$ steps to accept. Describe a *two-tape* Turing machine that recognizes the same language L, and takes time *linear* in the length of the input string to accept strings in L.

2 A word game

(14 points, 7 for each part) Here is a game played with bit strings. If the string begins with 1, remove the 1 and append 00 to the right end of the string. If it begins with 00, remove the 00 and append 10. If it begins with 01, remove the 01 and append 11. Thus if you run this game for several steps, starting with 1, you get

$$1 \to 00 \to 10 \to 000 \to 010 \to 011 \to 111 \to 1100 \to \cdots$$

Let L be the set of all bit strings that can be derived from 1 in this manner. (So, for example, every string in the example above is in L.)

(a) Show that L is Turing-recognizable (that is, describe an algorithm that semi-decides L).

(b) Show that L is actually decidable (describe an algorithm that decides it).

3 A reduction...

(10 points) Let $\mathcal{M}$ be a Turing machine and $w \in \{0,1\}^*$. Describe how to construct a new Turing machine $\mathcal{M}'$ such that $\mathcal{M}'$ accepts w and *no other bit string* if $\mathcal{M}$ accepts w, and $\mathcal{M}'$ accepts *no bit string at all* if $\mathcal{M}$ does not accept w.

4 ...and its consequences

(16 points, 2 for each correct answer) Now consider the following four decision problems, described as languages.

$$L_{TM} = \{< \mathcal{M}, w >: \mathcal{M} \text{ accepts } w\}.$$

$$EMPTY = \{< \mathcal{M} >: \mathcal{M} \text{ recognizes the empty language }\}.$$

$$NONEMPTY = \{< \mathcal{M} >: \mathcal{M}\text{accepts some bit string } \}.$$

$$UNIQUE = \{< \mathcal{M} >: \mathcal{M} \text{ accepts exactly one bit string}\}.$$

Tell whether the following statements are true or false. You may use the fact proved in class that L_{TM} is Turing-recognizable but not Turing-decidable.

(a) The construction in Problem 3 is a reduction of L_{TM} to $EMPTY$.

(b) ...a reduction of L_{TM} to $NONEMPTY$.

(c) ...a reduction of L_{TM} to $UNIQUE$.

(d) ...a reduction of $NONEMPTY$ to L_{TM}.

(e) This reduction proves that $EMPTY$ is undecidable.

(f) This reduction proves that L_{TM} is undecidable.

(g) This reduction proves that $UNIQUE$ is undecidable.

(h) This reduction proves that $EMPTY$ is not Turing-recognizable.

Incidentally, *all* of the assertions about decidability and recognizability in (e)-(h) are true, but you are being asked if the reduction in 3 proves the assertion.