Test 2

CSCI2244-Randomness and Computation

April 4, 2019

Figure 1: A giant jar of jellybens

Make sure to show all your work. In the second part, carefully identify the events and conditional probabilities in your solution. The problems call for exact numerical answers *except* in 2(c,d), where you may leave an expression that can be evaluated through use of a normal calculator by someone who knows nothing about probability. In cases where you can't complete a calculation all the way to a numerical answer, you may leave an unevaluated expression, for partial credit.

1 Diving for Jellybeans, Part 1

Visitors to the jellybean factory are invited to plunge an arm into a two-foot tall jar of jellybeans and grab a handful of beans.[1]

Some people do not reach very far at all into the jars and daintily pick out a few beans near the top. Others go in past the elbow to try to get to the bottom of the jar. But most are somewhere in between. Let X be the random variable whose value is the depth, in feet, to which a random visitor to the factory reaches. (Thus $X = 2$ means the bottom of the jar, $X = 0$ the top of the jar.) After some study, researchers determine that the density of X has the form

$$f_X(t) = \begin{cases} 0 & : t < 0 \\ c \cdot (4t - t^3) & : 0 \leq t \leq 2 \\ 0 & : 2 < t \end{cases},$$

for some $c > 0$. The graph of f_X is shown below. Observe that there is no scale on the vertical axis, so that you cannot determine the correct value of c from the picture itself.

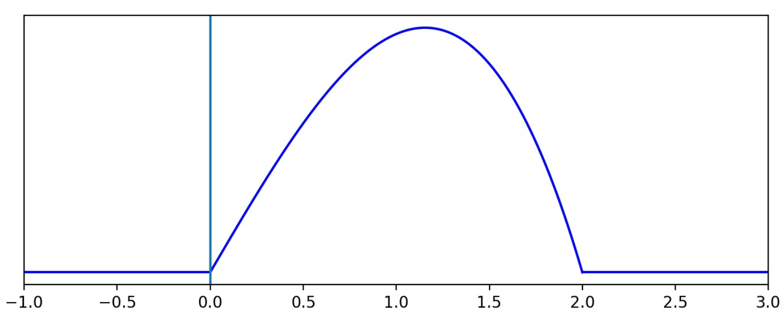

Figure 2: The density function of X.

[1] Try not to think about the sanitary implications.

(a) Determine the value of c.

Solution. We need

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= c\int_0^2 (4x - x^3)dx \\ &= c(2x^2 - \frac{x^4}{4})\Big|_0^2. \\ &= c(8-4) \\ &= 4c, \end{aligned}$$

so $c = \frac{1}{4}$.

(b) Determine the cumulative distribution function of X, and make a sketch of its graph.

Solution. From the calculation above,

$$F_X(t) = \begin{cases} 0 & : t < 0 \\ \frac{t^2}{2} - \frac{t^4}{16} & : 0 \le t \le 2 \\ 1 & : 2 < t \end{cases}.$$

The plot is shown in the figure below.

(c) Determine the probability that a random visitor reaches into the *bottom* half of the jar.

Solution. This asks for

$$P(X > 1) = 1 - F_X(1) = 1 - (1/2 - 1/16) = 9/16.$$

(d) A large number of people visit the jellybean factory during the course of a year and reach into the jar. What, approximately, is the average depth to which they reach?

Solution. This asks for

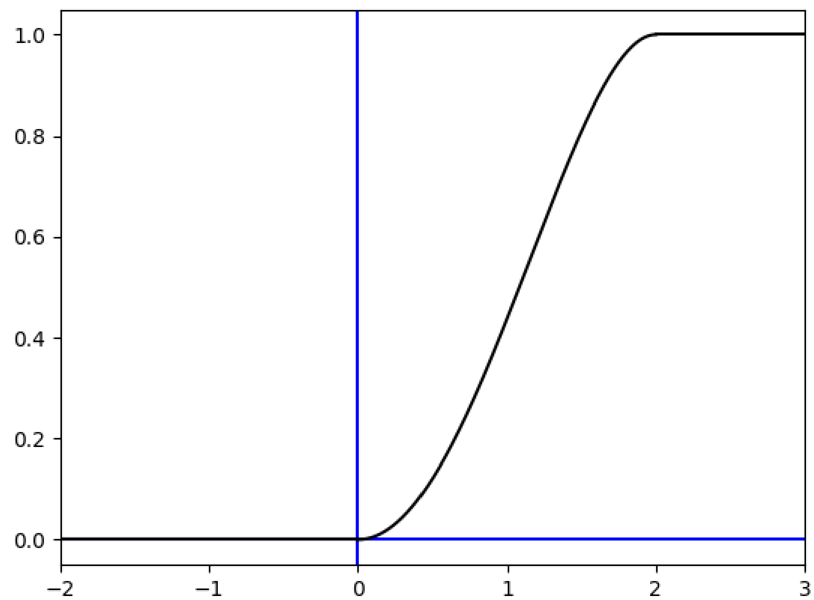

Figure 3: The cumulative distribution function of X.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_0^2 (x^2 - x^4/4) dx \\ &= (\frac{x^3}{3} - \frac{x^5}{20})\Big|_0^2 \\ &= \frac{8}{3} - \frac{32}{20} \\ &= \frac{16}{15}. \end{aligned}$$

(e) The *median* depth is the depth d such that $P(X < d) = \frac{1}{2}$. As it turns out, the median of X is quite close to 1. Is it greater than 1 or less than 1?

Solution.

$$P(X < 1) = F_X(1) = 1/2 - 1/16 < 1/2,$$

so the median is greater than 1. Note that you do not actually have to compute the median here!

2 Diving for Jellybeans, Part 2

For (a) and (b) below, you should give an exact value, expressed as a fraction. For (c), you can given an unevaluated expression. The correct solution to (d) uses the value from (c), which you can simply denote as C in your answer.

(a) Approximately 50% of the many thousands of jellybeans in the jar are red. You reach in and pull out 6 beans. What is the probability that at least four of them are red?

Solution. This is sampling without replacement, but because there are many thousands of jellybeans, you can model it as sampling with replacement, which is just the binomial distribution with a $p = \frac{1}{2}$—that is, a fair coin toss. The probability of at least four red beans is thus

$$2^{-6}(\binom{6}{4} + \binom{6}{5} + \binom{6}{6}) = 2^{-6}(15 + 6 + 1) = \frac{22}{64}.$$

(b) Someone else reaches in and pulls out 6 beans. Without showing them to you, she announces that at least four of them are red. What is the probability (assuming that she is not lying) that all 6 of them are red?

Solution. We are asked to compute the conditional probability.

$$P(\text{all 6 red}|\text{at least 4 red}).$$

This is just

$$\frac{P(\text{all 6 red})}{P(\text{at least 4 red})} = \frac{1}{64}/\frac{22}{64} = \frac{1}{22}.$$

A new exhibit at the visitor center in the factory features a row of three jellybean jars, each containing different concentrations of red jellybeans. (See the figure, and never mind the fact that the posted concentrations don't really match the picture!) Visitors are invited to choose one of the jars and to reach in and grab 6 jellybeans at random. It is found that visitors will choose the middle jar 45% of the time, the left-hand jar 25% of the time, and the right-hand jar the remaining 30% of the time.

(c) What is the probability that a random visitor picks 6 red jellybeans?

Solution. This is an application of the 'law of total probability'. Let L, C, R denote, respectively, the events that the visitor picks the left, center, or right jar. Then

$$\begin{aligned}
P(\text{all 6 red}) &= P(\text{all 6 red}|L)P(L) + P(\text{all 6 red}|C)P(C) + P(\text{all 6 red}|R)P(R) \\
&= 0.2^6 \times 0.25 + 0.5^6 \times 0.45 + 0.8^6 \times 0.3 \\
&= 0.0857.
\end{aligned}$$

(You didn't need to compute the last line.)

(d) A visitor picks 6 red jellybeans. What is the probability that he picked them from the right-hand jar?

Solution. This is an application of Bayes's Theorem.

$$\begin{aligned}
P(R|\text{all 6 red}) &= \frac{P(\text{all 6 red}|R)P(R)}{P(\text{all 6 red})} \\
&= \frac{0.8^6 \times 0.3}{0.0857} \\
&= 0.9178
\end{aligned}$$

Figure 4: Three jellybean jars all in a row.