CSCI2244-Randomness and Computation
Second Exam

April 12, 2018

Each part of each problem is worth 10 points, for 100 total points.

Show all your work. Most of the problems have a numerical answer, and in many cases you are actually given the value of the answer: this will give you a way to check that your derivation is correct, and enable you to use the value in subsequent parts of the problem even if you don't succeed in finding it yourself. You do not have to calculate the numerical values—in other words, it is perfectly acceptable to give an answer in a form like:

$$\prod_{j=0}^{5}\left[\binom{20}{\lceil \log_5 j \rceil} \cdot e^{\sqrt{20.5}}\right],$$

to give a perfectly preposterous example. On the other hand, you should not leave integral or derivative signs in your final answers.

1 Hit the target

Some children play a game throwing tomatoes at a vertical blue stripe painted on a white fence. Figure 1 shows the result of their efforts. The stripe is 6 inches = 0.5 feet wide. The horizontal positions on the fence are co-ordinatized in feet, with the center of the stripe as $x = 0$, so that the stripe runs from $x = -0.25$ to $x = 0.25$. In this problem we do not care about the vertical co-ordinate.

Let X be a random variable giving the horizontal co-ordinate of a tomato thrown by the best player, whose name is Champ. After some study, it is determined that X is well modeled by a density function of the form

$$P_X(x) = \begin{cases} 0, & |x| > 2 \\ C \cdot (4 - x^2), & |x| \leq 2 \end{cases},$$

where $C > 0$ is a constant. The graph is shown in Figure 2. (This figure is not to scale, so you cannot figure out anything about C from the picture.) By the way, the symmetry in this graph is your friend; use it wisely, and you can cut out about half your work.

(a) Find the value of C. (The answer is about 0.094.)

(b) What is the cumulative distribution function of X? (Give a formula for this and a rough sketch of its graph.)

(c) What is the probability that a tomato thrown by Champ strikes the target? For purposes of this problem, assume that the tomato is a single point, and that hitting the target means that the x-co-ordinate of the point is between -0.25 and 0.25. Y (The value is about 0.187.)

(d) What is the probability of Champ hitting the target at least 3 times in 10 throws? (The answer is about 0.28.)

(e) It's pretty easy to see that $E(X) = 0$. Find the *variance* of X.

(f) Let $Y = X_1 + \cdots + X_{25}$, where the X_i are pairwise independent random variables with the same distribution as X. Find the *standard deviation* of Y.

(g) Find $E(|X|)$. (There are several ways to do this—one of them is an 'Aha!' insight that lets you write down easily what the density of $|X|$ is.)

2 Light bulbs

The time to failure of a light bulb is modeled by the exponential distribution.[1] Recall that the density of the exponential distribution is given by

$$P_X(x) = \begin{cases} 0, & |x| < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases},$$

[1] Textbooks love to say this; I really don't believe it! But just pretend it's true.

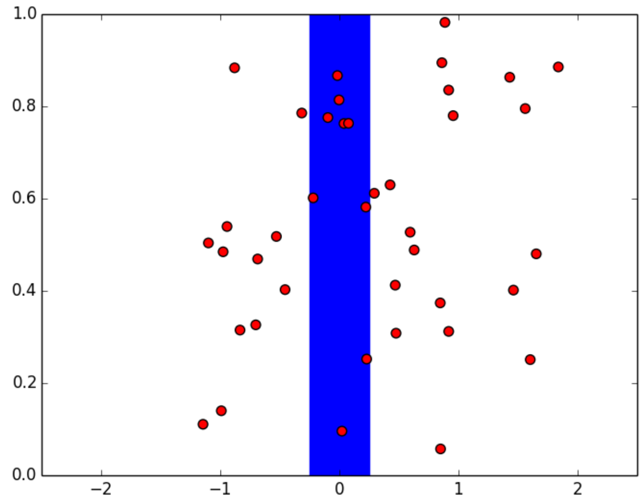

Figure 1: Ripe red tomatoes thrown against a wall. The blue stripe is half a foot wide, centered at $x = 0$.

where the average lifetime is $\frac{1}{\lambda.}$ Let us suppose that we have two different lots of light bulbs, one (Lot 1) with an average lifetime of 1000 hours, the other (Lot 2) with an average lifetime of 2000 hours.

(a) Find the probabilities that a light bulb from each of the lots functions for 1200 hours without burning out. (So there are two answers here; the probability for Lot 1 and the probability for Lot 2. The values are about 0.30 and 0.55.)

(b) Find the *median* lifetimes of bulbs from each of the lots. (The median of a random variable X is the value m such that $P(X < m) = \frac{1}{2}$. In other words, half the bulbs burn out before time m, and half after.)

(c) You pick a bulb at random from Lot 1 or Lot 2, without knowing which lot you chose; assume the probability of picking each lot is exactly one-half. You turn the bulb on, wait two thousand hours (!) and find that the bulb is still burning. What is the probability that you chose the bulb from Lot 1?

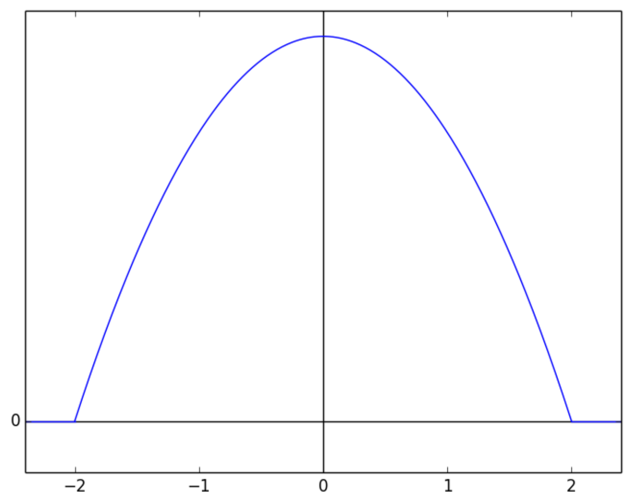

Figure 2: Density function of the horizontal coordinate of a tomato thrown at the wall.