CSCI2244-Randomness and Computation
First test

February 26, 2019

All told, there are ten problems, each worth ten points. In all cases, you should explain your reasoning carefully and identify by name any known distributions (*e.g.,* geometric, binomial, Poisson) whose properties you are applying. With the exception of Problem 1, where the arithmetic is pretty simple, you may leave your answer as an unevaluated expression involving binomial coefficients, powers, the exponential function, *etc.* For example, something like

$$\frac{\binom{6}{4} \cdot e^{-5}}{4^7}$$

is what I'm looking for. You can't really go much farther without a calculator or computer. (In fact, I have included the numerical answer with the problem, so your job is to supply the formula by which this answer was computed.)

All of the problems concern Skee-Ball, a classic arcade game invented in 1908 (thank you, Wikipedia). (See Figure 1). The player rolls a 3-inch diameter ball up a slightly inclined plane, trying to land the ball in one of the baskets shown, and is awarded points according to the values labeling the baskets.

Figure 1: *A vintage Skee-Ball machine (source: Joyous! at English Language Wikipedia)*

Of course, there is great variation in the skill of individual players. We will make our probability model of Skee-Ball based on lengthy observation of many rolls by a single player, shown in the table below.

Points	Frequency
0	38%
10	30%
20	15%
30	10%
40	5%
50	2%

1 A single roll.

Let X be the random variable whose value is the number of points received on a single roll.

(a) What is $P(X \geq 30)$?

Solution. We decompose the event into the disjoint union of three events:

$$\begin{aligned} P(X \geq 30) &= P(X = 30) + P(X = 40) + P(X = 50) \\ &= 0.1 + 0.05 + 0.02 \\ &= 0.17. \end{aligned}$$

(b) Sketch the graph of the PMF of X.

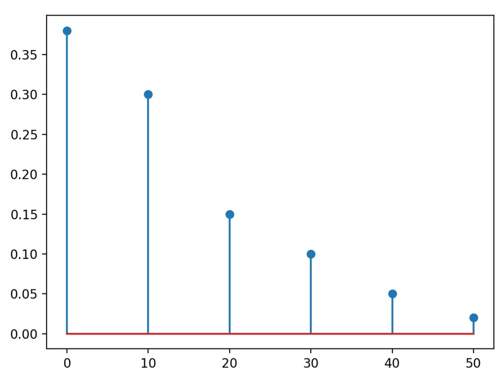

Figure 2: *Solution of Problem 1(b)*

(c) What is $E(X)$? Approximately how many points will the player receive if he makes 1,000 rolls?

Solution.

$$\begin{aligned} E(X) &= 0.38 \times 0 + 0.3 \times 10 + 0.15 \times 20 + 0.1 \times 30 + 0.05 \times 40 + 0.02 \times 50 \\ &= 3 + 3 + 3 + 2 + 1 \\ &= 12. \end{aligned}$$

So approximately 12,000 points in 1000 rolls.

2 A session with five rolls.

When you put a quarter in, you get five balls. We can model the outcome of 5-ball session as a quintuple of point values That is, the underlying sample space is

$$S = \underbrace{\{0, 10, 20, 30, 40, 50\} \times \cdots \times \{0, 10, 20, 30, 40, 50\}}_{\text{5 times}}.$$

(a) What is $|S|$? *(Answer: 7776.)*

Solution. 6^5

(b) What is the probability that the player never scores a single point in his 5 rolls? Describe this event as a subset E of S. Keep in mind that the probability distribution is not uniform, so that the answer is not $\frac{|E|}{|S|}$. *(Answer: 0.0079)*

Solution. $E = \{(0, 0, 0, 0, 0)\}$. That is, $|E| = 1$, but, as the reminder says, the answer is not $\frac{1}{|S|}$. Instead, we have 5 independent events, each occurring with probability 0.38, so the solution is 0.38^5.

(c) Let's say that a roll is *excellent* if it lands in one of the three baskets marked 30, 40, 50. What is the probability that the player makes *exactly* three excellent rolls? What is the probability that the player makes *at least* three excellent rolls? (See Problem 1(a) above). *(Answer: 0.0338,0.0375)*

Solution. This is a **binomial** distribution problem with $p = 0.17$. We are asking the probability of three successes in five throws, so this is

$$\binom{5}{3} \times 0.17^3 \times 0.83^2.$$

For the probability of at least 3 throws, the answer is

$$\binom{5}{3} \times 0.17^3 \times 0.83^2 + \binom{5}{4} \times 0.17^4 \times 0.83 + \binom{5}{5} \times 0.17^5.$$

(d) What is the probability that in the player's five rolls he gets all five nonzero scores—that is he scores a 10, a 20, a 30, a 40 and a 50, but not necessarily in that order? *(Answer: 0.00054)*

Solution. First consider the probability that the 5 rolls come up in the exact order 10,20,30,40,50. This is

$$0.38 \times 0.3 \times 0.15 \times 0.1 \times 0.05 \times 0.02.$$

The calculation will be the same for any one of the $5!$ permutations of these five scores, so the answer is

$$5! \times 0.38 \times 0.3 \times 0.15 \times 0.1 \times 0.05 \times 0.02.$$

3 Repeated rolls

In this problem state carefully any underlying properties of random variables that you use to solve the problem.
(a) Our player goes through all his quarters, and makes 100 rolls. What is the probability that he makes at least three 50-point rolls? You should give both an exact answer (which will involve binomial coefficients) and a good approximation (which will involve the exponential function). *(Answer: 0.32331,0.32332)*

Solution. The number of 50-point rolls in 100 rolls is a **binomial** random variable Y with PMF

$$P_Y(k) = \binom{100}{k} \times 0.02^k \times 0.98^{100-k},$$

so the probability of getting at least three 50-point rolls is:

$$\sum_{k=3}^{100} \binom{100}{k} \times 0.02^k \times 0.98^{100-k}.$$

However, it is far easier to use the complementary probability, which has only three terms instead of 98:

$$1 - (0.98^{100} + 100 \times 0.02 \times 0.98^{99} + \binom{100}{2} \times 0.02^2 \times 0.98^{98}).$$

This is well approximated by the **Poisson** distribution with $\lambda = 100 \times 0.02 = 2$. Again using the complementary probability, we get

$$1 - e^{-2}(1 + 2 + \frac{2^2}{2!}) = 1 - 5e^{-2}.$$

(b) Our player rolls repeatedly until making a roll that earns at least 10 points. Once this is achieved, he rolls again repeatedly until making at least 20 points. This continues, raising the goal to 30, then 40, then 50. How many rolls on average will it take to complete this task? (For example, rolling 20,10,30,40,50 does *not* complete the task, but rolling 20,10,30,40,50,50 does.) *(Answer: 74.9)*

Solution. Let Y be the random variable giving the number of rolls required to complete the task. We write

$$Y = Y_1 + Y_2 + Y_3 + Y_4 + Y_5,$$

where Y_i is the number of rolls required to complete the i^{th} task. Y_i is a **geometric** random variable with parameter p, where $p = 1 - 0.38 = 0.62$ when $i = 1$; $p = 1 - 0.38 - 0.3 = 0.32$ when $i = 2$;*etc.* Thus

$$\begin{aligned} E(Y) &= E(Y_1) + \cdots + E(Y_5) \\ &= \frac{1}{0.62} + \frac{1}{0.32} + \frac{1}{0.17} + \frac{1}{0.07} + \frac{1}{0.02}. \end{aligned}$$

4 The arcade supply closet

As the arcade manager, you are occasionally called upon to replace missing or damaged balls.

Figure 3: *Skee-Ball balls*

In your supply closet there is a box of 50 balls, 12 of which are the traditional hardwood balls, and 38 of which are plastic imitations.

You go into the closet and grab 5 balls at random without looking. What is the probability that exactly two of them are hardwood balls? *(Answer: 0.262)*

Solution. This is the basic **hypergeometric** distribution, with $N = 50, n = 5, K = 2$. The answer is

$$\frac{\binom{12}{2}\binom{38}{3}}{\binom{50}{5}}$$

5 How'd you do?

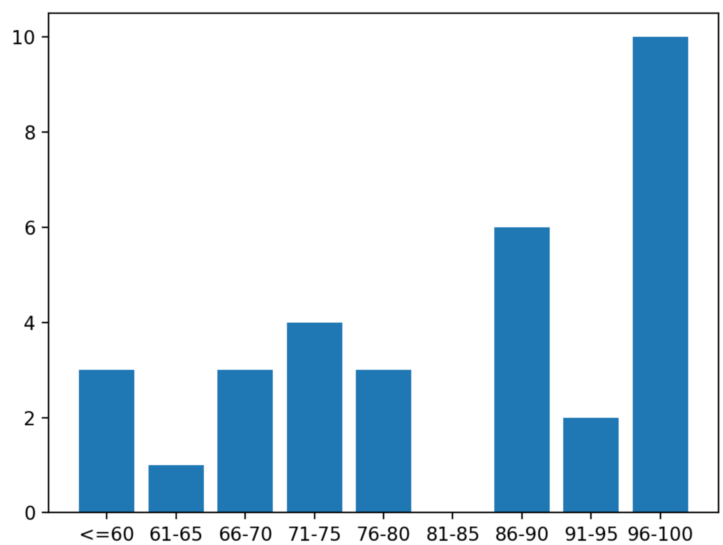

Figure 4: Histogram of exam results. The median was 88/100.