CSCI2244-Randomness and Computation
First Exam

March 1, 2018

Each part of each problem is worth 5 points. There are actually two parts to Problem 2, since you are asked to compute two probabilities. Problems 1 and 4 require numerical answers, except as indicated, but as always, you have to show your work: If the right answer is 7, then just '7' is the wrong answer! The calculations are entirely rudimentary, involving arithmetic with one-digit integers for the most part, so you do not need to use a calculator.

1 A Little Box of Chocolates

A little box of chocolates contains five chocolate truffles: two dark chocolates–the kind I like–and three milk chocolates, the kind I don't care for. I close my eyes grab two of the truffles at random.

(a) Model the outcomes of the experiment as the set S of two-element subsets of $\{1, 2, 3, 4, 5\}$, where 1 and 2 represent the dark chocolate truffles. What is $|S|$? (Give an exact numerical answer, as well as an expression involving binomial coefficients.)

(b) Let E be the event 'at least one of the chocolates I pull out is a dark chocolate'. Write the event E explicitly as a subset of S. What is $|E|$?

(c) Assume that each outcome in S has equal probability. What is $P(E)$?

2 A Bigger Box of Chocolates

Same problem, scaled up: A bigger box of chocolates contains twenty truffles, 8 of which are dark chocolate, and 12 of which are milk chocolate. I reach in and pull out ten truffles. What is the probability that *exactly* five of them are dark chocolate? What is the probability that *at least* five of them are dark chocolate? Your answer should contain binomial coefficients and summations, but you do not need to evaluate these.

3 The Chocolate Factory

The Chocolate Factory makes millions of chocolate truffles every day. Approximately one of every thousand chocolates that leave the factory contains a golden ticket. You order a shipment of 2500 chocolates.

(a) What is the probability that your shipment contains NO golden tickets? Give a (nearly) exact answer, based on treating this as a problem of sampling with replacement, along with an easy-to-compute approximation.

(b) What is the probability that your shipment contains at least two golden tickets? Just write the easy-to-compute approximation.

(c) You repeat this experiment many times, and keep track of how many golden tickets are in each shipment. What is the average number of golden tickets that you get in a shipment? Justify your answer—you can cite any result that was taught in class.

4 I couldn't figure out how to make this problem about chocolate, so it's about dice.

Roll two fair 6-sided dice. Let the random variable X_1 denote the number of spots showing on the first die, and let X_2 be the number of spots showing on the second die. Let $Z = |X_1 - X_2|$; that is, Z is the absolute value of the *difference* between the two dice. (For instance, $Z = 5$ if you roll a 1 and a 6.)

(a) Formally, Z is a real-valued function on the sample space $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$. What is $Z((2, 4))$?

(b) What is $P(Z = 0)$? What is is $P_Z(1)$? What is $F_Z(2)$? (Recall that P_Z denotes the PMF of Z, and F_Z the CDF of Z.)

(c) Determine all the values of P_Z and sketch the graph of this function. (The preceding problem will get you halfway there.)

(d) Find $E(Z)$. You can leave the answer as an unevaluated expression. (The numerical answer is about 1.944, which should help tell you if you did the problem correctly.)

(e) Let W be the random variable denoting the larger of the values on the two dice. That is

$$W = \max(X_1, X_2).$$

Determine $E(W)$, by applying the handy fact

$$\max(x, y) = \frac{x + y + |x - y|}{2}.$$

You can use the answer to the preceding question, even if you did not solve it.