CS244
Final Exam
May 9, 2014

1. This is a game played with a shuffled deck of cards. Remember that there are 52 cards, divided into 4 suits with 13 cards in each suit. In this game only the suits matter. You pull three cards (without replacement) from a shuffled deck and **win if the cards are from three different suits**

(a) Describe a sample space Ω and probability function P appropriate to this experiment. You should do this in such a manner that all the outcomes in Ω are equally likely. You can use unevaluated expressions involving binomial coefficients and factorials in your answer.

(b) What is the probability that you win at the game? Write an exact expression for this probability. Your answer can include unevaluated factorials and binomial coefficients. The answer is approximately 0.398.

2. In a variation of the game, you **win if all three suits are the same**.

(a) What is the probability of winning at this game. Once again you should write an exact expression for this probability as in 1(b). ? (The numerical answer is approximately 0.052.)

(b) You play this version of the game 50 times. Determine the probability that you win at least twice. Write an expression for this probability using binomial coefficients and the numerical answer given in 2(a).

(c) Now write an approximate, but easier-to-evaluate expression for the probability in (b) using the exponential function. (When evaluated, the result in (b) turns out to be about 0.74 when rounded to two digits, and the result in (c) about 0.73.)

3. In yet another variation, you are awarded points depending on the number of distinct suits among the three cards you draw: You get 3 points if all the suits are the same (the scenario of Problem 2), 2 points if all the suits are different (the scenario of Problem 1) and 1 point if there are exactly two distinct suits. Let X be a random variable denoting the number of points you receive on a play of this game.

(a) Determine E(X). (Use the numerical values given in Problems 1 and 2 to write an expression for E(X). You don't need to do all the arithmetic to evaluate it. The numerical answer is approximately 1.501)

(b) Determine Var(X) and the standard deviation of X. (The numerical answer for the standard deviation is approximately 0.595)

4. You play the game described in Problem 3 one hundred times. Estimate the probability that the total number of points is more than 160. In your answer, you can use the cumulative standard normal distribution function
Φ as well as the numerical results in Problem 3.

5. Now consider the roll of a standard 6-sided die. You have learned the alarming news that approximately 20% of the dice in use have been altered so that 6 comes up with probability 0.1, and each of the other numbers with probability 0.18.

You choose a die at random and roll it 100 times, getting 11 6's. E denotes the event `you get 11 6's on 100 rolls'.

(a) Write expressions (you don't need to evaluate them) for

p= P[E|the die is fair]
q =P[E|the die is altered]

(b) What is the probability that the die has been altered? You can write your answer in terms of p and q above.

6. You choose a point uniformly and at random inside a circle of radius 1.
Let X denote the distance of the point from the center.

(a) Determine formulas for both the CDF and the PDF of X, and sketch plots of both these functions.

(b) Determine E(X).

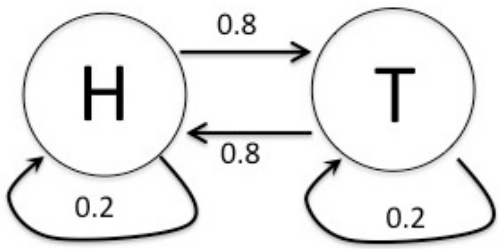
(c) Determine $E(X^2)$. There are (at least) two ways to go about this, either by using the PDF of X that you determined in (a), or by finding the PDF of X^2 and working directly with that.

7. Consider the matrix

$$M = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$$

(a) Show that $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$ is an eigenvector of M. What is the associated eigenvalue?
(b) Find an eigenvector of M that is associated with a different eigenvalue. (HINT. If you use the fact that M is symmetric then you can do this with hardly any computation.) What is the associated eigenvalue?

You repeatedly flip a magical coin. The coin does not like to repeat itself: so if the coin comes up heads on one toss, the probability that it will be heads on the next toss is 0.2 (and 0.8 that the next toss will be tails). Similarly, if the coin comes up tails on one toss, the probability that it will come up tails on the following toss is 0.2. The behavior is summarized in the diagram below.



(c) Suppose the coin came up heads on the first toss. What is the probability that it comes up heads on the third toss?

(d) Assume again that the coin came up heads on the first toss. What is the approximate probability that the coin comes up heads on the 100th toss? Justify your answer carefully, using facts about eigenvalues and stationary distributions for Markov chains.