Assignment 6: Continuous Sample Spaces

CSCI244-Randomness and Computation
Assigned March 14
Due Wednesday, March 27

March 20, 2019

1 Monte Carlo Integration

A short fat right circular cone whose base has radius 1, centered at $(0, 0, 0)$, with apex at $(0, 0, 1)$, has been stabbed through the heart by a long skinny right circular cone whose base has radius $\frac{1}{2}$ centered at $(-2, 0, \frac{1}{2})$ and apex $(2, 0, \frac{1}{2})$. The crime scene is depicted in Figure 1.

Estimate the volume of the *intersection* of these two cones.

The details. The short fat cone is the set of points

$$\{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 1 - \sqrt{x^2 + y^2}\}.$$

The long skinny cone is the set of points

$$\left\{(x, y, z) : (z - 1/2)^2 + y^2 \leq \frac{1}{4}, -2 \leq x \leq 2 - 8\sqrt{y^2 + (z - 1/2)^2}\right\}.$$

The idea in Monte Carlo integration is to generate a lot of points uniformly at random in a solid S that contains the figure F that you are interested in, and whose volume you know. The probability that such a point lies in F is

$$\frac{\mathrm{vol}(F)}{\mathrm{vol}(S)}.$$

So if you generate N points in S and M of those points lie within the figure F, we have

$$\frac{M}{N} \approx \frac{\mathrm{vol}(F)}{\mathrm{vol}(S)}.$$

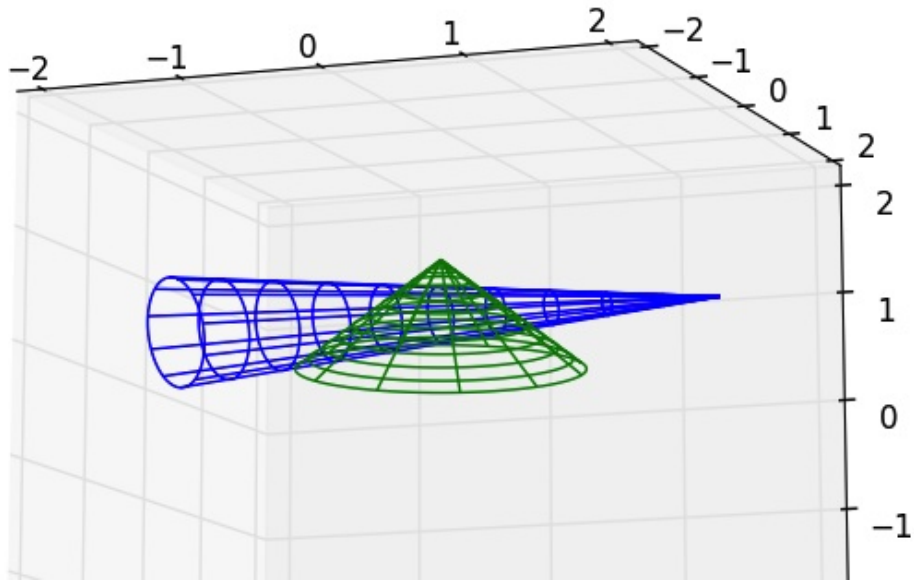

Figure 1: One cone passing through another

Part of the trick is to choose a figure S that fits F reasonably closely, so as to reduce the error in the approximation. Describe how you did this.

Food for thought. Still, Monte Carlo integration is not terribly accurate. Can you find the volume exactly? This seems like a hard calculus problem, and I don't know how to do it off the top of my head, but I would bet that it's possible.

It's also hard to figure out exactly what the intersection looks like! Can you produce accurate drawings of the figure? Make one on a 3D printer?

2 Break a Stick, Make a Triangle

The problem: Calculate exactly the probability of various events in the sample space modeling a pair of spinners. Use this to answer a question about breaking sticks, and compare to the results of a simulation.

The details. Let's start with our two spinners that generate a pair of numbers between 0 and 1. Remember that we can model the set of outcomes as the points in a 1×1 square with the uniform probability distribution, so that the probability of an event is equal to the area of the corresponding region of the square. Use this

to answer the following questions. You will probably want to draw pictures of the regions in question–you can do this using matplotlib's `fill` function, but if you prefer you can use any other painting or drawing software you like, or even scan hand-drawn figures.
(a) What is the probability that the value on at least one of the spinners is less than $\frac{1}{2}$?
(b) What is the probability that the values on the two spinners are within $\frac{1}{2}$ of each other?
(c) What is the probability that the values on the two spinners are equal?
(d) What is the probability of *(a)* and *(b)* together—that is at least one value is less than $\frac{1}{2}$ and the two values are within $\frac{1}{2}$ of each other?
(e) Break a stick one inch long in two different places. Assume that the two places x and y where the stick is broken are chosen at random between 0 and 1. What is the probability that we can form a triangle from the three pieces that result? (In order to make a triangle with the three pieces, each piece must have length less than $\frac{1}{2}$. So we have to choose x and y so that x and y differ by no more than one-half, and so that at least one of the numbers is less than one-half and one of them more than one-half. So you can use some of the results from (a)-(d) here.)
(f) Now write a simulation of the stick-breaking experiment and confirm that the experimental results match the probability obtained in (e).

3 Darts

Once again, our sample space is the set of points on a dart board one foot in diameter, which we model as the disk

$$\{(x, y) : x^2 + y^2 \leq 1\}.$$

We have already computed the density function and the cumulative distribution function of the random variable given by the distance of the dart from the target, under the assumption that the dart hits are uniformly distributed in the disk.

Let us suppose instead that there is a different probability distribution. Specifically, we are in the company of talented darts players who are more likely than not to come close to the target, so that the graph of the density function of the random variable is the straight line segment in Figure 2. We call this random variable X.

(a) What is the y-coordinate of the point at which this at line intersects the y-axis? Explain.

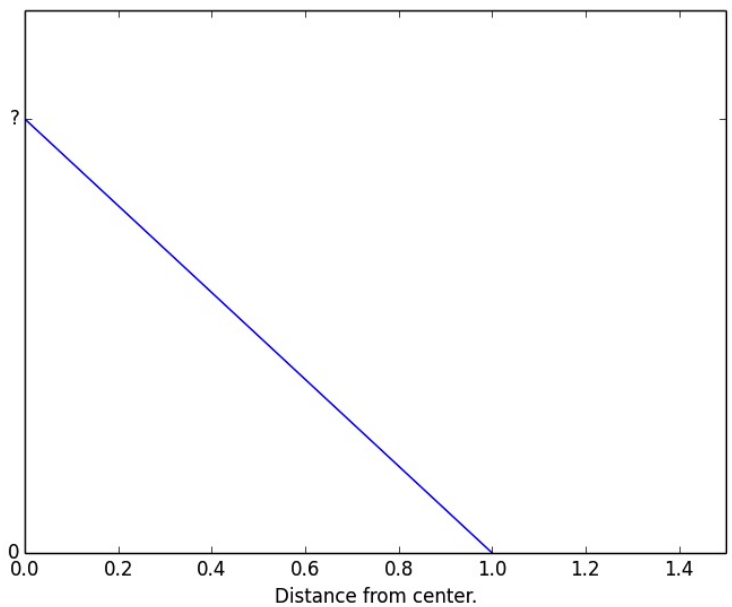

Figure 2: Probability density function for distance of dart from center

(b) What is the probability that a dart lands within 1 inch ($1/12$, since the distance on the x-axis is given in feet) of the target?

(c) Determine a distance d so that approximately half the darts hit within a distance of d feet of the target. (This is the *median* of the random variable X.)

(d) Find a formula for the cumulative distribution function and plot it.

(e) Determine $E(X)$.

Each of these problems has *two* challenges: correctly modeling the problem, and then dredging up enough calculus to find the answer.

4 Some harder problems, for extra credit

4.1 Break a stick, make a triangle, revisited

The story is the same as 2(e), but now the breaking strategy is to first choose a single random break point, and then break the *larger* of the two resulting pieces at a random spot. (HINT: This is much harder problem, and I admit I got it completely wrong when I first tried it and discovered that my answer did not agree with a simulation. The big insight here is that you are still generating two numbers between 0 and 1 independently and at random, so the sample space is still the unit square with the uniform distribution, but the event 'the pieces form a triangle' has changed.) Run a simulation and make sure that your answer agrees with it.

4.2 Buffon checkerboard

In class and in the lecture notes, we solved the Buffon needle problem, where we imagined a floor on which horizontal lines are spaced one inch apart, and a one-inch needle is tossed onto the floor. We found that the probability that the needle crosses a line is $\frac{2}{\pi}$. Here I ask the same question except now there are both horizontal and vertical lines spaced one inch apart, making a checkerboard pattern of 1×1 squares. What is the probability that the needle crosses a line? (Obviously it should be larger than before.)