Assignment 4

CSCI2244-Randomness and Computation
Due Wednesday, February 20, at 11:59PM

Some problems about discrete random variables.

1 Roll of three dice.

Let Y_1, Y_2, Y_3 be mutually independent random variables each with the uniform distribution on $\{1, \ldots, 6\}$. Let $Z = Y_1 + Y_2 + Y_3$. In other words, Z represents the sum of three fair dice.

This problem is quite cookbook-ish, since it requires only a few small tweaks to the posted code that does the same thing for two dice. (I will demonstrate this in class on February 12.) Pay attention to the shape of the PMF—it should look almost familiar, and it will look even more familiar if you try to do the same thing for *four* dice.

1.1 PMF

Compute and plot the PMF of Z. (HINT: Work smart, not hard. The posted code showing the same problem for two dice gives a model to follow. It begins by computing the underlying sample space, which is the cartesian product $\{1, \ldots, 6\} \times \{1, \ldots, 6\}$. You only have to add one more `for` to the initial list comprehension to get the set of all triples. Your code should include a function that returns a list of the values of the PMF, and a second function that creates the plot.)

1.2 Simulation using CDF

Write a function `dice3_sim(n)` that returns a list of n values with the same distribution as Z. You should do this in such a way that generating each value requires only a single call to `random()`; so instead of simulating the rolls of three individual dice, you will used the CDF of Z. Again, the model to follow is the posted code that solves the same problem for two dice. The trick is to apply the built-in numpy function `searchsorted` to find the index of the smallest value of the CDF that is greater than or equal to the value returned by `random()`.

1.3 Expected value of Z.

Find $E(Z)$. Then use your simulation to compute several thousand samples from Z and take their average. You should get very close agreement.

2 Poisson distribution

This problem is taken from the book by Grinstead and Snell. A professor in the mathematics department receives on average 10 letters every working day. He assumes that the number of letters he receives each day has the Poisson distribution.

2.1 Basic probability calculations with the Poisson distribution

What is the probability that he receives exactly ten letters on a given day? What is the probability that he receives fewer than five letters? You can calculate the numerical values using just a few lines typed into the Python shell, but you should describe the calculations in your writeup.

2.2 A day with no mail.

A day with no mail is a rare occurrence. What is the probability that on a given day the professor receives no mail? The professor wants to know how likely this is to occur over a long period of time. He goes to collect his mail 300 days each year. What is the probability that in ten years there will be a day with no mail? (This second question is very similar to the one asked in Problem 1.1 of the preceding assignment.)

3 Hypergeometric distribution

You pull thirteen cards from a shuffled deck.

3.1 Probability of getting a certain number of spades

What is the probability that exactly 5 of the cards are spades? That at least 5 of the cards are spades? (There are 13 spades in the complete deck.)

3.2 Most likely number of spades.

What is the most likely number of spades?

3.3 Expected number of spades.

What is the expected number of spades?

4 Cowrie shells and Pachisi

Pachisi is a popular board game played in India (see https://en.wikipedia.org/wiki/Pachisi). It is similar to Backgammon, in that the players' pieces race around the board. A westernized version was marketed under the name Parcheesi. In Backgammon and Parcheesi, dice are used to determine how many spaces a piece can move, however in the traditional Indian game, the players throw six cowrie shells, and count the number of shells that land with the opening facing up. The pieces then advance according to the following table:

Number of shells facing up	Number of spaces to move
0	25
1	10
2	2
3	3
4	4
5	5
6	6

Figure 1: Yes, I really did go out and buy some cowrie shells. Here I threw six of them onto a chair in my office; two of them are facing up.

4.1 Number of shells facing up.

Let X denote the number of shells facing up on a single throw of six shells. Find the PMF of X. You will need to know the probability that a shell lands facing up. Assume that this probability is 0.4.

4.2 Expected number of spaces on one turn: easy version.

Let Y denote the number of spaces a player advances on a given turn. Determine $E(Y)$.

4.3 Expected number of spaces on one turn: hard version.

This is a hard problem that I could not at first figure out when I thought of it (although the hint below will be a big help). So let's call this extra credit. I did not tell you one other rule: If a player throws 0,1 or 6, they advance the number of spaces given in the table, and get to throw again. I assume that this continues for as long as the player continues to throw 0, 1 or 6, so in principle there is no upper bound on the number of spaces a player can advance on a signle turn. Let Y' denote this new random variable. Find $E(Y')$. (HINT: Let Y_i denote the number of spaces that a player advances on the i^{th} throw of a turn, and use linearity of expectation. If i is large then Y_i will be 0 with high probability, because there will be fewer than i throws.)

5 Independent random variables

A 'theory' problem. In this problem, the underlying experiment is two successive tosses of a fair coin. Let the random variable X have the value 1 if the first toss is heads and 0 if it is tails Let Y be the same thing defined in terms of the second toss. Let $Z = |X - Y|$.

5.1

Give a simple description in words of Z, and find its PMF.

5.2 Pairwise independence is not the same as mutual independence.

X and Y, of course, are independent Show that X and Z are independent, and that Y and Z are independent, but that X, Y, Z are not mutually independent. (To show independence for X and Z, you have to tabulate the probabilities for all four cases, but you can leave out the calculation for Y and Z, since it is the same. To show non-independence for all three, you only need to give a single counterexample.)