Counting

- Counting
  - It seems easy enough:
    - 1, 2, 3, 4, ...
    - This approach works well for counting simple things like your fingers or complicated things for which there is no identifiable structure.

- However,
  - How do you count the number of different ways to select a dozen doughnuts when there are five varieties available?
  - How do you count the number of 16-bit numbers with exactly 4 ones?
  - ...
Counting

- Every counting problem determine the size of some set.
- **REMEMBER**: The size or cardinality of a set is the number of elements in the set.
Basic counting

- Counting problems may be very hard and not obvious

**Strategy**
- Simplify the solution by decomposing the problem

- Two basics decomposition rules:
  - **Product rule**
  - **Sum rule**

- These rules serve as the foundation for almost all counting techniques
Linus allocates his sister Lucy a quota of 20 crabby days, 40 irritable days, and 60 generally surly days.

How many days can Lucy be out-of-sort one way or another?

Let: \( C = \) crabby days, \( I = \) irritable days, \( S = \) surly days.

So the answer to the question is: \(|C \cup I \cup S|\)

NOTE: Lucy is permitted at most one bad quality each day.
Sum rule

- **Sum rule**: If $A_1, A_2, \ldots, A_n$ are disjoint sets, then
  
  $$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

- Thus, Lucy can be out-of-sort for:
  
  $$|C \cup I \cup S| = |C| + |I| + |S| = 20 + 40 + 60 = 120$$
Sum rule

- If there are 24 CS major, 17 ECE major, none of them are double major.
- How many ways to pick a representative who is either CS or ECE major?
Product rule

- A daily diet consist of a breakfast select from set $B$, a lunch selected from set $L$, and a dinner selected from set $D$.
  - $B = \{\text{pancakes, bacon and eggs, bagel, Doritos}\}$
  - $L = \{\text{burger and fries, salad, Doritos}\}$
  - $D = \{\text{pizza, burrito, pasta, Doritos, chicken}\}$

- How many different diets are possible?

- **REMEMBER:** Cartesian product

  $$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$
Product rule

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- How many different diets are possible?

- The set of all possible daily diets is $B \times L \times D$ e.g.:
  - (pancakes, burger and fries, pizza)
Product rule

- **Product rule:** If $A_1, A_2, \ldots, A_n$ are sets, then
  \[ |A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n| \]

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- How many different diets are possible?
  \[ |B \times L \times D| = |B| \cdot |L| \cdot |D| = 4 \cdot 3 \cdot 5 = 60 \]
Exercises

- A new company with just two employees, Bob and Carol, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

- The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
Exercises

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
Exercises

What is the value of \( k \) after the following code has been executed?

\[
\begin{align*}
k &:= 0 \\
\text{for } i_1 &:= 1 \text{ to } n_1 \\
&\quad \text{for } i_2 := 1 \text{ to } n_2 \\
&\quad \quad \ldots \\
&\quad \quad \text{for } i_m := 1 \text{ to } n_m \\
&\quad \quad k := k + 1
\end{align*}
\]
More complex counting

- **Password problem:** On a certain computer system, a valid password is a sequence of between six and eight symbols. The first symbol must be a letter (which can be uppercase or lowercase), and the remaining symbols must be either letters or digits.

- **How many different password are possible?**

- **Define two sets:**
  - \( F \) (valid symbols in first position) = \{a, b, ..., z, A, B, ..., Z\}
  - \( S \) (valid symbols in subsequent positions in the password) = \{a, b, ..., z, A, B, ..., Z, 0, 1, ..., 9\}
The inclusive-exclusive principle

- A computer company receives 350 applicants from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these people majored in computer science, 147 majored in business, and 51 majored both in computer science and in business.

- How many of these applicants majored neither in computer science nor in business?
The inclusive-exclusive principle

- Let $A$ and $B$ be sets:
  - There are $|A|$ ways to select an element from $A$, and
  - There are $|B|$ ways to select an element from $B$.

- The number of way to select an element from $A$ or from $B$ is:

$$|A \cup B| = |A| + |B| - |A \cap B|$$
The inclusive-exclusive principle

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- How many of these applicants majored neither in computer science nor in business?

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316 \]

- 316 is the number of students who majored either in computer science or in business

- 350 – 316 = 34 applicants majored neither in cs nor in business
The inclusive-exclusive principle

- How about $|A \cup B \cup C|$?

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|
\]
Counting problems can be solved using tree diagrams. A tree consists of:
- a root
- a number of branches leaving the root
- possible additional branches leaving the endpoints of other branches
- leaves

In counting:
- A branch represents a possible choice
- The possible outcomes are represented by the leaves
Suppose you have 4 shirts, 3 pairs of pants, and 2 pairs of shoes. How many different outfits do you have?