Basis step: \((0, 0) \in S\).

Recursive step: If \((a, b) \in S\), then \((a + 2, b + 3) \in S\) and \((a + 3, b + 2) \in S\).

a) List the elements of \(S\) produced by the first five applications of the recursive definition.
b) Use strong induction on the number of applications of the recursive step of the definition to show that \(5|a + b\) when \((a, b) \in S\).
c) Use structural induction to show that \(5|a + b\) when \((a, b) \in S\).

27. Let \(S\) be the subset of the set of ordered pairs of integers defined recursively by

Basis step: \((0, 0) \in S\).

Recursive step: If \((a, b) \in S\), then \((a, b + 1) \in S\), \((a + 1, b + 1) \in S\), and \((a + 2, b + 1) \in S\).

a) List the elements of \(S\) produced by the first four applications of the recursive definition.
b) Use strong induction on the number of applications of the recursive step of the definition to show that \(a \leq 2b\) whenever \((a, b) \in S\).
c) Use structural induction to show that \(a \leq 2b\) whenever \((a, b) \in S\).

28. Give a recursive definition of each of these sets of ordered pairs of positive integers. (Hint: Plot the points in the set in the plane and look for lines containing points in the set.)

a) \(S = \{(a, b) | a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a > 2, a + b \text{ is odd} \}
\)
b) \(S = \{(a, b) | a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a + b \text{ is even} \}
\)
c) \(S = \{(a, b) | a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a + b \text{ is odd and } 3 \mid b \}
\)

29. Give a recursive definition of each of these sets of ordered pairs of positive integers. Use structural induction to prove that the recursive definition you found is correct. (Hint: To find a recursive definition plot the points in the set in the plane and look for patterns.)

a) \(S = \{(a, b) | a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a + b \text{ is odd} \}
\)
b) \(S = \{(a, b) | a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a + b \text{ is even} \}
\)
c) \(S = \{(a, b) | a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, a + b \text{ is odd and } 3 \mid b \}
\)

30. Prove that in a bit string, the string 01 occurs at most one more time than the string 10.

31. Define well-formed formulae of sets, variables representing sets, and operators from \(\{\emptyset, \cup, \cap, \neg\}\).

32. a) Give a recursive definition of the function \(\text{ones}(s)\), which counts the number of ones in a bit string \(s\).
b) Use structural induction to prove that \(\text{ones}(st) = \text{ones}(s) + \text{ones}(t)\).

33. a) Give a recursive definition of the function \(m(s)\), which equals the smallest digit in a nonempty string of decimal digits.
b) Use structural induction to prove that \(m(st) = \min(m(s), m(t))\).

The reversal of a string is the string consisting of the symbols of the string in reverse order. The reversal of the string \(w\) is denoted by \(w^R\).

34. Find the reversal of the following bit strings.
a) 0101  b) 1011  c) 0100 0101 0111

35. Give a recursive definition of the reversal of a string. (Hint: First define the reversal of the empty string. Then write a string \(w\) of length \(n + 1\) as \(xy\), where \(x\) is a string of length \(n\), and express the reversal of \(w\) in terms of \(x^R\) and \(y\).)

36. Use structural induction to prove that \((w_1w_2)^R = w_2^Rw_1^R\).

37. Give a recursive definition of \(w^i\) where \(w\) is a string and \(i\) is a nonnegative integer. (Here \(w^i\) represents the concatenation of \(i\) copies of the string \(w\).)

38. Give a recursive definition of the set of bit strings that are palindromes.

39. When does a string belong to the set \(A\) of bit strings defined recursively by

\[
\lambda \in A \\
0x1 \in A \text{ if } x \in A,
\]

where \(\lambda\) is the empty string?

40. Recursively define the set of bit strings that have more zeros than ones.

41. Use Exercise 37 and mathematical induction to show that \(l(w^i) = i \cdot l(w)\), where \(w\) is a string and \(i\) is a nonnegative integer.

42. Show that \((w^R)^i = (w^i)^R\) whenever \(w\) is a string and \(i\) is a nonnegative integer; that is, show that the \(i\)th power of the reversal of a string is the reversal of the \(i\)th power of the string.

43. Use structural induction to show that \(n(T) \geq 2h(T) + 1\), where \(T\) is a full binary tree, \(n(T)\) equals the number of vertices of \(T\), and \(h(T)\) is the height of \(T\).

The set of leaves and the set of internal vertices of a full binary tree can be defined recursively.

Basis step: The root \(r\) is a leaf of the full binary tree with exactly one vertex \(r\). This tree has no internal vertices.

Recursive step: The set of leaves of the tree \(T = T_1 \cdot T_2\) is the union of the set of leaves of \(T_1\) and the set of leaves of \(T_2\). The internal vertices of \(T\) are the root \(r\) of \(T\) and the union of the set of internal vertices of \(T_1\) and the set of internal vertices of \(T_2\).

44. Use structural induction to show that \(l(T)\), the number of leaves of a full binary tree \(T\), is 1 more than \(i(T)\), the number of internal vertices of \(T\).

45. Use generalized induction as was done in Example 15 to show that if \(a_{m,n}\) is defined recursively by \(a_{0,0} = 0\) and

\[
a_{m,n} = \begin{cases} 
a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\
a_{m,n-1} + 1 & \text{if } n > 0,
\end{cases}
\]

then \(a_{m,n} = m + n\) for all \((m, n) \in \mathbb{N} \times \mathbb{N}\).
EXAMPLE 19 Suppose that ‘I Love New Jersey’ tee shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the tee shirt?

Solution: The tree diagram in Figure 4 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different tee shirts.

Exercises

1. There are 18 mathematics majors and 325 computer science majors at a college.
   a) How many ways are there to pick two representatives, so that one is a mathematics major and the other is a computer science major?
   b) How many ways are there to pick one representative who is either a mathematics major or a computer science major?

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

3. A multiple-choice test contains ten questions. There are four possible answers for each question.
   a) How many ways can a student answer the questions on the test if every question is answered?
   b) How many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

5. There are six different airlines that fly from New York to Denver and seven that fly from Denver to San Francisco. How many different possibilities are there for a trip from New York to San Francisco via Denver, when an airline is picked for the flight to Denver and an airline is picked for the continuation flight to San Francisco?

6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

7. How many different three-letter initials can people have?

8. How many different three-letter initials with none of the letters repeated can people have?

9. How many different three-letter initials are there that begin with an A?

10. How many bit strings are there of length eight?

11. How many bit strings of length ten begin and end with a 1?

12. How many bit strings are there of length six or less?

13. How many bit strings with length not exceeding \( n \), where \( n \) is a positive integer, consist entirely of 1s?

14. How many bit strings of length \( n \), where \( n \) is a positive integer, start and end with 1s?

15. How many strings are there of lowercase letters of length four or less?

16. How many strings are there of four lowercase letters that have the letter \( x \) in them?

17. How many strings of five ASCII characters contain the character @ (at sign) at least once? (Note: There are 128 different ASCII characters.)

18. How many positive integers less than 1000
   a) are divisible by 7?
   b) are divisible by 7 but not by 11?
   c) are divisible by both 7 and 11?
19. How many positive integers between 100 and 999 inclusive
a) are divisible by 7?
b) are odd?
c) have the same three decimal digits?
d) are not divisible by 4?
e) are divisible by 3 or 4?
f) are not divisible by either 3 or 4?
g) are divisible by 3 but not by 4?
h) are divisible by 3 and 4?

20. How many positive integers between 1000 and 9999 inclusive
a) are divisible by 9?
b) are even?
c) have distinct digits?
d) are not divisible by 3?
e) are divisible by 5 or 7?
f) are not divisible by either 5 or 7?
g) are divisible by 5 but not by 7?
h) are divisible by 5 and 7?

21. How many strings of three decimal digits
a) do not contain the same digit three times?
b) begin with an odd digit?
c) have exactly two digits that are 4s?

22. How many strings of four decimal digits
a) do not contain the same digit twice?
b) end with an even digit?
c) have exactly three digits that are 9s?

23. A committee is formed containing either the governor or one of the two senators of each of the 50 states. How many ways are there to form this committee?

24. How many license plates can be made using either three digits followed by three letters or three letters followed by three digits?

25. How many license plates can be made using either two letters followed by four digits or two digits followed by four letters?

26. How many license plates can be made using either three digits followed by three digits or four letters followed by two digits?

27. How many license plates can be made using either two or three letters followed by either two or three digits?

28. How many strings of eight English letters are there
a) if letters can be repeated?
b) if no letter can be repeated?
c) that start with X, if letters can be repeated?
d) that start with X, if no letter can be repeated?
e) that start and end with X, if letters can be repeated?
f) that start with the letters BO (in that order), if letters can be repeated?
g) that start and end with the letters BO (in that order), if letters can be repeated?
h) that start or end with the letters BO (in that order), if letters can be repeated?

29. How many strings of eight English letters are there
a) that contain no vowels, if letters can be repeated?
b) that contain no vowels, if letters cannot be repeated?
c) that start with a vowel, if letters can be repeated?
d) that start with a vowel, if letters cannot be repeated?
e) that contain at least one vowel, if letters can be repeated?
f) that contain exactly one vowel, if letters can be repeated?
g) that start with X and contain at least one vowel, if letters can be repeated?
h) that start and end with X and contain at least one vowel, if letters can be repeated?

30. How many different functions are there from a set with 10 elements to sets with the following numbers of elements?
a) 2 b) 3 c) 4 d) 5

31. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?
a) 4 b) 5 c) 6 d) 7

32. How many functions are there from the set \{1, 2, \ldots, n\}, where \(n\) is a positive integer, to the set \{0, 1\}?

33. How many functions are there from the set \{1, 2, \ldots, n\}, where \(n\) is a positive integer, to the set \{0, 1\}
a) that are one-to-one?
b) that assign 0 to both 1 and \(n\)?c) that assign 1 to exactly one of the positive integers less than \(n\)?

34. How many partial functions (see the exercises in Section 1.8) are there from a set with five elements to sets with each of these number of elements?
a) 1 b) 2 c) 5 d) 9

35. How many partial functions (see the exercises in Section 1.8) are there from a set with \(m\) elements to a set with \(n\) elements, where \(m\) and \(n\) are positive integers?

36. How many subsets of a set with 100 elements have more than one element?

37. A palindrome is a string whose reversal is identical to the string. How many bit strings of length \(n\) are palindromes?
Exercises

1. Show that in any set of six classes there must be two that meet on the same day, assuming that no classes are held on weekends.

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
   a) How many socks must he take out to be sure that he has at least two socks of the same color?
   b) How many socks must he take out to be sure that he has at least two black socks?

4. A bowl contains ten red balls and ten blue balls. A woman selects balls at random without looking at them.
   a) How many balls must she select to be sure of having at least three balls of the same color?
   b) How many balls must she select to be sure of having at least three blue balls?

5. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

6. Let $d$ be a positive integer. Show that among any group of $d + 1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by $d$.

7. Let $n$ be a positive integer. Show that in any set of $n$ consecutive integers there is exactly one divisible by $n$.

8. Show that if $f$ is a function from $S$ to $T$ where $S$ and $T$ are finite sets with $|S| > |T|$, then there are elements $s_1$ and $s_2$ in $S$ such that $f(s_1) = f(s_2)$, or in other words, $f$ is not one-to-one.

9. What is the minimum number of students, each of whom comes from one of the 50 states, enrolled in a university to guarantee that there are at least 100 who come from the same state?

10. Let $(x_i, y_i), i = 1, 2, 3, 4, 5,$ be a set of five distinct points with integer coordinates in the $xy$ plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

11. Let $(x_i, y_i, z_i), i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, be a set of nine distinct points with integer coordinates in $xyz$ space. Show that the midpoint of at least one pair of these points has integer coordinates.

12. How many ordered pairs of integers $(a, b)$ are needed to guarantee that there are two ordered pairs $(a_1, b_1)$ and $(a_2, b_2)$ such that $a_1 \mod 5 = a_2 \mod 5$ and $b_1 \mod 5 = b_2 \mod 5$?

13. a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.
   b) Is the conclusion in part (a) true if four integers are selected rather than five?

14. a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.
   b) Is the conclusion in part (a) true if six integers are selected rather than seven?

15. How many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

16. How many numbers must be selected from the set {1, 3, 5, 7, 9, 11, 13, 15} to guarantee that at least one pair of these numbers add up to 16?

17. A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle location in the aisle, and shelf. There are 50 aisles, 85 horizontal locations in each aisle, and 5 shelves throughout the warehouse. What is the least number of products the company can have so that at least two products must be stored in the same bin?

18. Suppose that there are nine students in a discrete mathematics class at a small college.
   a) Show that the class must have at least five male students or at least five female students.
   b) Show that the class must have at least three male students or at least seven female students.

19. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.
   a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.
   b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

20. Find an increasing subsequence of maximal length and a decreasing subsequence of maximal length in the sequence 22, 5, 7, 2, 23, 10, 15, 21, 3, 17.

21. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.

22. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

23. Describe an algorithm in pseudocode for producing the largest increasing or decreasing subsequence of a sequence of distinct integers.

24. Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.
7. Find the number of 5-permutations of a set with nine elements.

8. In how many different orders can five runners finish a race if no ties are allowed?

9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

11. How many bit strings of length ten contain
   a) exactly four 1s?
   b) at most four 1s?
   c) at least four 1s?
   d) an equal number of 0s and 1s?

12. How many bit strings of length 12 contain
   a) exactly three 1s?
   b) at most three 1s?
   c) at least three 1s?
   d) an equal number of 0s and 1s?

13. A group contains \( n \) men and \( n \) women. How many ways are there to arrange these people in a row if the men and women alternate?

14. In how many ways can a set of two positive integers less than 100 be chosen?

15. In how many ways can a set of five letters be selected from the English alphabet?

16. How many subsets with an odd number of elements does a set with ten elements have?

17. How many subsets with more than two elements does a set with 100 elements have?

18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
   a) are there in total?
   b) contain exactly three heads?
   c) contain at least three heads?
   d) contain the same number of heads and tails?

19. A coin is flipped ten times where each flip comes up either heads or tails. How many possible outcomes
   a) are there in total?
   b) contain exactly two heads?
   c) contain at most three tails?
   d) contain the same number of heads and tails?

20. How many bit strings of length ten have
   a) exactly three 0s?
   b) more 0s than 1s?
   c) at least seven 1s?
   d) at least three 1s?

21. How many permutations of the letters ABCDEFG contain
   a) the string BCD?
   b) the string CFGA?
   c) the strings BA and GF?
   d) the strings ABC and DE?
   e) the strings ABC and CDE?
   f) the strings CBA and BED?

22. How many permutations of the letters ABCDEFGH contain
   a) the string ED?
   b) the string CDE?
   c) the strings BA and FGH?
   d) the strings AB, DE, and GH?
   e) the strings CAB and BED?
   f) the strings BCA and ABF?

23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? (Hint: First position the men and then consider possible positions for the women.)

24. How many ways are there for ten women and six men to stand in a line so that no two men stand next to each other? (Hint: First position the women and then consider possible positions for the men.)

25. One hundred tickets, numbered 1, 2, 3, \ldots, 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if
   a) there are no restrictions?
   b) the person holding ticket 47 wins the grand prize?
   c) the person holding ticket 47 wins one of the prizes?
   d) the person holding ticket 47 does not win a prize?
   e) the people holding tickets 19 and 47 both win prizes?
   f) the people holding tickets 19, 47, and 73 all win prizes?
   g) the people holding tickets 19, 47, 73, and 97 all win prizes?
   h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?
   i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?
   j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

26. Thirteen people on a softball team show up for a game.
   a) How many ways are there to choose ten players to take the field?
   b) How many ways are there to assign the ten positions by selecting players from the 13 people who show up?
   c) Of the 13 people who show up, three are women. How many ways are there to choose ten players
to take the field if at least one of these players must be a woman?

27. A club has 25 members.
   a) How many ways are there to choose four members of the club to serve on an executive committee?
   b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club?

28. A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

29. How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers in the correct order
   a) where consecutive means in the usual order of the integers and where these consecutive integers can perhaps be separated by other integers in the permutation?
   b) where consecutive means both that the numbers be consecutive integers and that they be in consecutive positions in the permutation?

30. Seven women and nine men are on the faculty in the mathematics department at a school.
   a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?
   b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

31. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain
   a) exactly one vowel?
   b) exactly two vowels?
   c) at least one vowel?
   d) at least two vowels?

32. How many strings of six lowercase letters from the English alphabet contain
   a) the letter a?
   b) the letters a and b?
   c) the letters a and b in consecutive positions with a preceding b, with all the letters distinct?
   d) the letters a and b, where a is somewhere to the left of b in the string, with all the letters distinct?

33. Suppose a department contains ten men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

34. Suppose a department contains ten men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

35. How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

36. How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

37. How many bit strings of length ten contain at least three 1s and at least three 0s?

38. How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

39. How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

40. How many ways are there to seat six people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

41. How many ways are there for a horse race with three horses to finish if ties are possible? (Note: Two or three horses may tie.)

42. How many ways are there for a horse race with four horses to finish if ties are possible? (Note: Any number of the four horses may tie.)

43. There are six runners in the 100-yard dash. How many ways are there for three medals to be awarded if ties are possible? (The runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals.)

44. This procedure is used to break ties in games in the championship round of the World Cup soccer tournament. Each team selects five players in a prescribed order. Each of these players takes a penalty kick, with a player from the first team followed by a player from the second team and so on, following the order of players specified. If the score is still tied at the end of the ten penalty kicks, this procedure is repeated. If the score is still tied after 20 penalty kicks, a sudden-death shootout occurs, with the first team scoring an unanswered goal victorious.

   a) How many different scoring scenarios are possible if the game is settled in the first round of ten penalty kicks, where the round ends once it is impossible for a team to equal the number of goals scored by the other team?

   b) How many different scoring scenarios for the first and second groups of penalty kicks are possible if the game is settled in the second round of ten penalty kicks?

   c) How many scoring scenarios are possible for the full set of penalty kicks if the game is settled with no more than ten total additional kicks after the two rounds of five kicks for each team?